Foundations of Physics, Vol. 21, No. 7, 1991

Interpretation of the Curious Results of the New Quantum Formalism of Pre- and Post-Selected Systems

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Received September 18, 1990; revised February 6, 1991

The analysis, with the use of two state vectors, of a quantum system, during the time interval between two measurements, leads to some amazing results, which seem to contradict our usual "quantum common sense." We explore the questions of compatibility with the conventional quantum theory, uniqueness of pre- and post-selected ensembles, commutativity, simultaneity and reality of strong and weak values in the intermediate time, and the meaning of the weak value. Common criticisms are shown to be unfounded.

1. INTRODUCTION

In a series of papers Aharonov, Albert, and Vaidman (AAV)[(1–6)] used a time-symmetric formalism of quantum mechanics[(7)] for the description of quantum systems in the time interval between two measurements. Between the pre- and the post-selection (PPS) measurements, the system is described by two state vectors: The state resulting from the pre-selection measurement evolves from the past, and the state resulting from the post-selection measurement evolves backward in time from the future (of the intermediate time). Their analysis led to some intriguing results. They showed that the pre- and post-selected states can define a set of noncommuting variables, such that the result of a measurement of either one of them in the intermediate time can be ascertained with probability unity.[(2)] By "ascertain" we refer to the ability to predict the result one

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0015-9018/91/0700-0803$06.50/0

would get if the intervening measurement were done under the specified selection conditions. Throughout this paper we will use the word "ascertain" in this sense. AAV also introduced a new concept, which attracted much attention,[8-17] of weak measurements in the intermediate time: Consider the initial state of the measuring device to be a Gaussian of q with a spread Δ,

$$\langle q \mid MD(t=0)\rangle \propto \exp[-(q/2\Delta)^2] \tag{1}$$

Now the initial total state of the measuring device and the measured system will be $|\psi_1\rangle \mid MD(t=0)\rangle$. Let the interaction Hamiltonian be

$$H_{\text{in}}(t) = -g(t)\, qA, \qquad g(t) = \begin{cases} 1/T, & 0<t<T \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Assuming zero free Hamiltonian, the time-evolution operator for this state will be

$$U(T) = \exp\left[-i \int_0^T H_{\text{in}}(t)\, dt\right] \tag{3}$$

Adding the post-selection requirement that the system be at the state $|\psi_2\rangle$ at the end of the measurement, we will get the final state of the measuring device[2]

$$\langle\psi_2|\exp\left[-i\int_0^T H(t)\,dt\right]|\psi_1\rangle \exp[-(q/2\Delta)^2]$$

$$= \langle\psi_2|\exp(-iqA)\,|\psi_1\rangle \exp[-(q/2\Delta)^2] \tag{4a}$$

$$= \langle\psi_2|\psi_1\rangle \left\{\exp(-iqA_w) + \sum_{n=2}^{\infty} \frac{(iq)^n}{n!}\,[(A^n)_W - (A_w)^n]\right\} \exp[-(q/2\Delta)^2] \tag{4b}$$

where

$$A_w = \frac{\langle\psi_2|\,A\,|\psi_1\rangle}{\langle\psi_2|\psi_1\rangle} \tag{5}$$

As we can see from Eq. (4a), the total effect of the measurement is expressed in $\exp(iqA)$ (Appendix A). If the sum term in Eq. (4b) is neglected, then we will get that the total effect of the measurement is $\exp(iqA_w)$. The condition for neglecting the sum term is *the weakness condition*:

$$(2\Delta)^n \frac{\Gamma(n/2)}{(n-2)!}\,|(A^n)_w - (A_w)^n| \ll 1, \qquad \text{for all} \quad n \geqslant 2 \tag{6}$$

Thus, the result of a weak measurement of an observable A is the weak value A_w.

These weak values possess some fascinating properties. First, the weak value A_w is not bounded by the spectrum of A. Second, they are all simultaneously measurable. Third, these unexpected results are reproducible.

In this paper we examine some of the fundamental problems in interpreting AAV's work, and common criticisms[8] are shown to be unfounded. We show the source of the compatibility with the conventional quantum theory (ec. 2). We introduce a proof of the conservation of the probability transition of a PPS ensemble under specific intervening measurements (Appendix B), which is significant in the discussion of the uniqueness of PPS ensembles (Sec. 3). We explore commutativity, simultaneity and reality of weak measurements (Sec. 4). We conclude with the discussion of the physical meaning of weak values (Sec. 5).

2. COMPATIBILITY WITH THE CONVENTIONAL QUANTUM THEORY

AAV have shown and stressed[2,3] the complete agreement of the results of the two-vector formalism (TVF) with the conventional formalism. This is because the TVF is rooted in the conventional formalism. Let us describe three consecutive measurements:

a. A first measurement at t_1, after which the system is at the state $|\psi_1\rangle$.

b. An intermediate measurement at t of the observable A—$A\,|a_i\rangle = a_i\,|a_i\rangle$.

c. A second measurement at t_2, after which the system is at the state $|\psi_2\rangle$.

In the conventional description the probability for the transition from the state $|\psi_1\rangle$ to the state $|\psi_2\rangle$ via the state $|a_i\rangle$ is

$$P_i = |\langle\psi_2|\, U(t_2, t)\, |a_i\rangle\langle a_i|\, U(t, t_1)\, |\psi_1\rangle|^2 \tag{7}$$

Note that it can also be written in the TVF form:

$$P_i = |\langle U^+(t_2, t)\, \psi_2\, |a_i\rangle\langle a_i|\, U(t, t_1)\, \psi_1\rangle|^2 \tag{8}$$

This TVF form expresses an important change in our description of the measurements (Fig. 1): Here $|\psi_2\rangle$ evolves backwards in time toward the

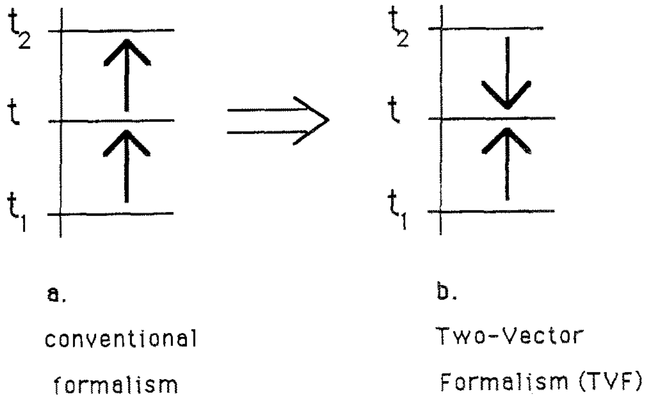

Fig. 1. A schematic description of the conceptual change suggested by Aharonov *et al.* for the analysis of a quantum system in the time interval between two measurements.

intermediate measurement (which also implies that the collapse of the wave function due to a measurement evolves in both directions of time symmetrically). Still, the results of the TVF and those of the conventional formalism are identical.

There was no motivation to look for the TVF results in the conventional formalism. For example, in the conventional formalism the weak value is due to a peculiar state of the measuring device. This state is a special superposition of the measuring device's eigenstates, which interfere destructively except for the improbable weak value's region[2] (Fig. 2). In the TVF the weak value is derived by the same simple formula and reasoning as the usual strong values.

Hu complains that "the mathematical derivation of AAV obscures the actual physics of the (measurement) process."[8] He takes it for granted that the correct description is given by the conventional quantum formalism. We believe that in the light of the TVF's mathematical simplicity and its introduction of new questions, one should seriously consider the TVF as describing "the actual physics of the process." In addition, the TVF enables us to write new states which are superpositions of selections,[2,4] called *generalized states*:

$$\sum c_i \langle \psi_i | \, | \phi_i \rangle$$

This opens up new possibilities. For example, the determination of σ_x, σ_y, and σ_z for the intervening measurement can be done with the use of such states.[4] Note that the generalized states cannot be written directly in the conventional formalism.

Usually, physicists hesitate to admit results which depend on superpositions of a macroscopic body (such as the measuring device). Therefore, in the conventional single-vector formalism, finding a spin component equal to 100 for a spin-1/2 particle would be considered a meaningless statistical error. In the TVF such a result would be the necessary outcome of the weak measurement performed on a PPS system. The additional information involved in the TVF gives us a direct interpretation in terms

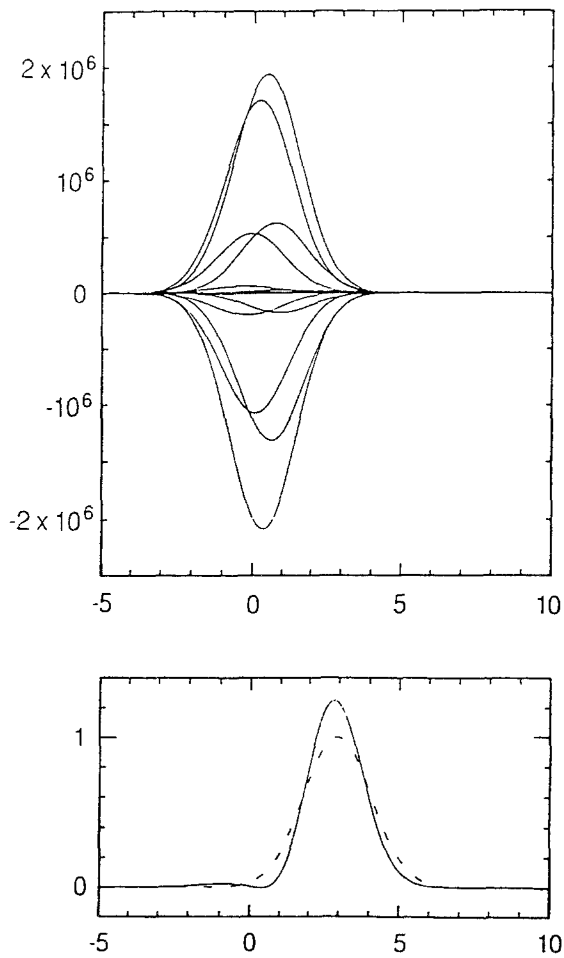

Fig. 2. Superposition of Gaussians shifted by values between −1 and 1 with the appropriate phase differences equal to a single Gaussian shifted by the value 3 (from Aharonov and Vaidman[2]).

of predictable results, instead of statistical errors. Thus the picture supplied by the conventional formalism is less heuristic compared with the TVF, when analyzing PPS systems.

3. UNIQUENESS OF THE PPS ENSEMBLE

Hu[8] follows Bub and Brown[9] with the claim that the PPS measurements do not define a unique ensemble: "The identity of the ensemble depends upon the measurement taken in the intervening period," since "a particular system which is pre- and post-selected under one intervening measurement may not necessarily be under another."[8] It is true that if an ensemble was pre- and post-selected without any strong intermediate measurement, then we cannot tell for each of the individual systems whether it would remain a part of the PPS ensemble if an intermediate measurement were performed. Yet even in the conventional quantum mechanics it is not clear which statements can be made for individual quantum systems.

In their context of analysis of the PPS ensemble, what AAV[2,9] claim is that the TVF is a correct way to analyze the result of the second in a given series of three consecutive measurements, which were all done in the past. In this context AAV can infer retrodictively in the following sense: If you tell me which observable was measured (strongly) in the intervening period, then sometimes I can specify the exact result, and in any case I can say more about the intervening state than I can say using only pre-selection. This is because including the future boundary condition of the system makes available additional information for the analysis at the intermediate time.

Hu goes further to conclude that "one cannot infer retroductively the determinateness of noncommuting observables."[8] Yet, we cannot accept Hu's argument as general for there is at least one case in which the identity of the ensemble clearly does not change under an intervening measurement: the case in which the intervening strong measurement commutes with either of the boundary measurements. For example, when an ensemble is pre-selected for $|\sigma_x = +1\rangle$ and post-selected for $|\sigma_y = +1\rangle$ the non-disturbing intervening measurement could be either σ_x or σ_y. Each of these possible measurements would not change the identity of the ensemble, because of its time-symmetric definition. Note that one can ascertain the outcomes of these specific noncommuting intervening measurements. The subtle problem is that one cannot ascertain that the inferences are made for the same individual systems, because in one PPS procedure we can include only one strong intervening measurement.

It was overlooked so far that when one can ascertain the result of the intervening measurement (for generalized states one can consider intervening measurement which does not commute with either boundary measurement[4]), the probability transition from the pre- to the post-selected state does not change because of the intervening measurement (appendix B). Thus, the necessary condition for a unique definition of a PPS ensemble, that the overall transition probability is conserved, is satisfied for the specific set of observables, *the ascertainable set*, whose measurements can be ascertained for this ensemble. This is analogous to the conventional case in which we perform a measurement which commutes with the one that defined the ensemble. Therefore Hu's arguments against the ability to ascertain the value of noncommuting variables are not conclusive. Note that our arguments do not contradict Bub and Brown's criticism[9] that "the notion of a statistical ensemble which is specified by pre-selection and post-selection via an *arbitrary* intervening measurement is not well defined in quantum mechanics... without specification of the intervening measurement."

To conclude, there is an interesting set of variables (even non-commuting) for which the result of measuring one of them in the intermediate period can be ascertained. The overall transition probability is the same for all of these cases, which *a priori* enables the participation of exactly the same systems in all these PPS ensembles. But practically, because of the limitation of performing only one strong intervening measurement, one cannot prove or refute that the inferences are made for the same individual systems.

If one is analyzing within the context of the TVF, then there is a different sense in which the definition of the ensemble changes by an intervening strong measurement. The intervening strong measurement separates the time interval into two subintervals: The result of the intervening measurement is projected forward and backward in time, thus the initial state is confined to the time interval between the pre-selection and the intervening measurement, while the final state is confined between the post-selection and the intervening measurement (Fig. 3).

In the case of a weak intervening measurement, which does not disturb the initial or the final state significantly (Appendix C), the definition of the PPS ensemble (by the pre- and post-measurements alone) does not change in any of the senses considered above. Thus, contrary to what Hu argues, there is a unique physical description of a quantum system during the time interval between two measurements in terms of the weak values.

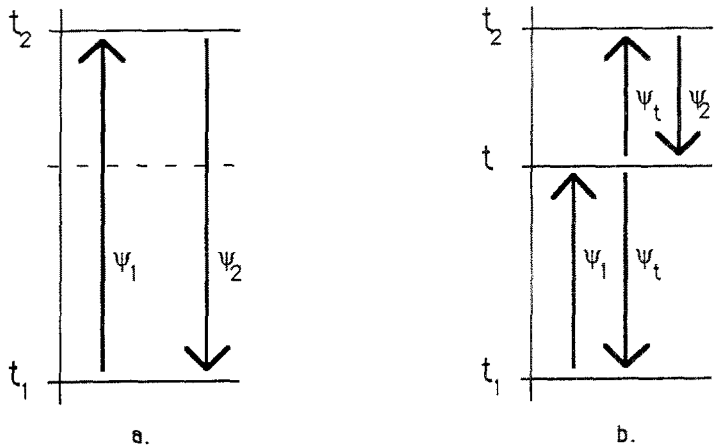

Fig. 3. In a system which is pre- and post-selected, both the pre- and the post-selected states exist in the intermediate time (Fig. 3a). The addition of an intervening measurement seperates the intervening time into two periods (Fig. 3b). This is not the case when the intervening measurement is weak. In such cases the boundary-selected states penetrate, unaltered, through the intervening weak measurement (symbolized by the dashed line in Fig. 3a).

4. COMMUTATIVITY OF WEAK MEASUREMENTS AND SIMULTANEITY OF WEAK VALUES

The commutation relations and the uncertainty principle can be interpreted as two faces of the same coin. The first expresses the importance of the order of measurements, while the second stresses the disturbance of the state by a measurement. In his discussion of the question "in what way can we say the weak measurements commute?" Hu tries to attack the claim for nondisturbance in order to refute the commutativity of weak measurements. In his words: "It does not seem surprising that by sacrificing preceision... we reobtain some degree of commutativity. However, if we view measurement disturbance as the fundamental problem..." Hu analyzes the case of spin measurements: pre-selection of $|\sigma_x = +1\rangle$, post-selection of $|\sigma_y = +1\rangle$, and two intermediate weak measurements of σ_y and σ_x in this order. He shows that after the weak σ_y measurement there is a resulting down component (of σ_y) that follows through all the way to the final state $|\sigma_y = +1\rangle$. From this he concludes that "it remains that the ensemble is fundamentally disturbed by the extra intervening measurement of σ_x which, like the precise measurement, opened up other possible paths to the final state."

To counter this argument, we first remark that no system was in the $|\sigma_y = -1\rangle$ state during the measurements. They were all in the same pure state that was insignificantly slanted from the $|\sigma_x = +1\rangle$ direction (Appendix C). The central point is that the author takes the "mathematician's view" that if the two states are not identical, then they are "fundamentally disturbed," while Aharonov *et al.* take the "physicist's view" that the almost identical states can be regarded as "fundamentally similar." This conflict touches on the very real issue of how we interpret approximations, which are commonly used in all branches of physics. We believe that the physical interpretation supports Aharonov's approach, which is also illustrated in the following result: For any two variables, even noncommuting $[A, B] \neq 0$, we define $C = A + B$. The result of a weak measurement will yield $C_w = A_w + B_w$.[2] This is the additivity property of the weak values. The weak values of noncommuting variables exist simultaneously in the sense that their simultaneous measurements will yield the same results as if they were measured separately.

Isn't there a contradiction here with the uncertainty principle? No, weakening the interaction reduces the accuracy of the measurements.[16] In our experiments the error in the result of a single weak measurement is greater than the minimal uncertainty due to the Heisenberg principle, so that $\Delta A \cdot \Delta B \gg h$.

From the additivity of weak values we can derive a semiclassical description. Consider the weak value of the spin vector[2] σ_w: Pre-selection of $|\sigma_x = +1\rangle$ and post-selection of $|\sigma_\alpha = +1\rangle$ define two given projections of the total σ_w in the intermediate time. By complete analogy with the classical algebra of vectors, we can construct $\sigma_w(x, \alpha)$, the projection of σ_w on the (x, α) plane (Fig. 4a). Now the weak value in any direction in that plane can be immediately calculated as the projection of $\sigma_w(x, \alpha)$. Hu's

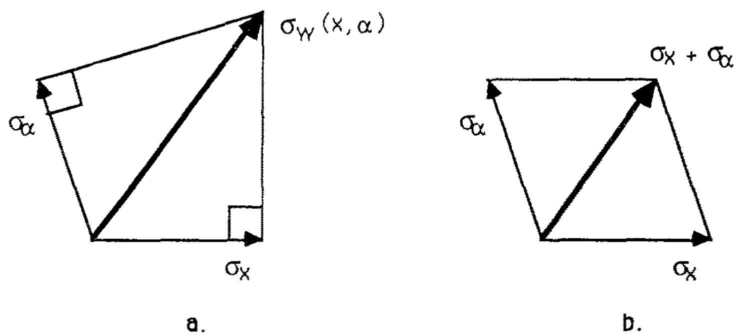

Fig. 4. Given two projections of a spin vector, σ_x and σ_α, one can construct the projection of that spin vector $\sigma_w(x, \alpha)$ in the plane (x, α) as in Fig. 4a. $\sigma_w\ (x, \alpha)$ is not equal to the vector addition of σ_x and σ_α which is constructed in Fig. 4b.

criticism of this semiclassical property results from a simple mistake: He takes the boundary values as two different vectors (like two given forces) instead of two projections of the same vector. When he calculates σ_w as their vectorial addition, he obviously gets the wrong value (Fig. 4b).

The simultaneous existence of the boundary-condition states (and therefore values) can be inferred indirectly through the weak values

$$A_w = \frac{\langle \psi_2 | A | \psi_1 \rangle}{\langle \psi_2 | \psi_1 \rangle}$$

The weak measurement is the measurement we can perform in the intermediate time in which both states manifest themselves simultaneously. We believe that the simultaneous existence of the boundary states in the intermediate time cannot be avoided.

5. THE MEANING OF THE WEAK VALUE

AAV declare that, "the most important outcome of our approach is the possibility to define a new concept—the weak value of a quantum system between two measurements, i.e., a property of a system belonging to an ensemble which is both pre-selected and post-selected. This property can manifest itself through a measurement which fulfills certain requirements of weakness."(2) AAV define a "weak value of a quantum variable" and a "strong value of a quantum variable," not a new kind of variable. Thus Hu's question, "what is the validity of the claim that σ_w represents a new quantum variable called the weak variable?"(8) is misleading.

Hu fails to see the real novelty of the weak-value concept. For instance, he has problems with the fact that the weak value "depends on a weakness condition for the measuring device."(8) However, understanding the weak value as the result of a weak measurement leads us to its physical meaning: "The effect of any interaction which is weak enough will depend on such weak values."(2) The weak value is the property that dominates weak measurements on a system. The weakness condition [Eq. (3)] gives an upper bound, and thus the range of the weak measurement lies from this limit down to a zero interaction (between the measuring device and the system). The weak value is universal in the sense that it is the same all over the range of weak interaction. We cannot claim that the weak value describes any usual quantum quantity of the system (a spin-1/2 fermion does not turn into a spin-100 boson(5)), yet it describes the behavior of the system under weak interaction.

The weak value A_w of the variable A is identical for every system belonging to the same PPS ensemble. The result of a weak measurement performed on a single quantum system has a big spread (thus yielding almost no information), but we can theoretically reduce it by a factor of $1/\sqrt{N}$ in two ways:

a. By concidering an ensemble of N systems, each selected seperately (repeating the same pre- and post-selections and intermediate weak measurements on N systems).

b. By performing one weak measurement on a PPS ensemble consisting of N systems. The probability to find such a PPS ensemble is $|\langle\psi_2|\psi_1\rangle|^{2N}$, which makes it a rare event. Here the weakness condition is for the interaction of each individual system with the measuring device.

For the ensemble, the weak value can be sharp as we want, and it can be regarded as a measurable macroscopic property of the ensemble. The idea can be illustrated by considering average operators such as $J = (1/N)\sum_{k=1}^{N} \sigma^k$. The interaction Hamiltonian is $H_{\text{in}}(t) = -g(t)q\sum_{k=1}^{N}(1/N)\sigma^k$ and its action on each seperate system is weakened by the factor $1/N$.[2] Since the weakness of the interaction is evident in the operator itself, it is not surprising that the commutativity of weak measurements follows naturally: Though $[\sigma_x^k, \sigma_y^k] = i\sigma_z^k \neq 0$, it still holds that $[J_x, J_y] = i(1/N)J_z \to 0$. Regarding the average operator as measuring a sharp macroscopic property of the ensemble is a common interpretation, such as in the case of measuring the magnetization of a single-domain ferromagnet with a compass' needle.

In contrast with strong intermediate measurements, more than one weak measurement can be performed without altering the PPS ensemble. They are not only simultaneously ascertainable but are also simultaneously measurable. Thus, the weak values fulfill the EPR criterion for physical reality.

To stress the meaning in which the system does possess the property of the weak value, it is important to note that the weak value satisfies the philosophical reality condition by the method of counterfactual implications. Namely, of a system belonging to a given PPS ensemble, one can say the following: If you had measured weakly the quantity A of the system, you would have obtained the value A_w. This holds without restrictions on the number or the order of the weak measurements performed in the intermediate period.

If the weak values will be observed in experiments,[17] it will have important implications on the validity of the Von Neumann formulation of the measurement process (Appendix A). The weak value is an unusual result arising directly from Von Neumann's theory. In a strong measurement, in order to find only one of the possible eigenvalues, Von Neumann imposed the collapse of the wave function to an eigenstate corresponding to the measured eigenvalue. In weak measurements there is no restriction on the state of the system after the measurement, since the result in the measuring device is unambiguous (unlike the multiple Gaussians obtained in strong measurements). Further discussion of the relations between the existence of weak values and the collapse of the wave function will be given in a following paper.

6. CONCLUSIONS

We noted the source of the complete agreement between the results of the conventional quantum formalism and those of the two-vector formalism (TVF), and argued that the actual physics of a pre- and post-selected (PPS) system in the intermediate time (between these two selection measurements) might be described by the TVF and not unconditionally by the conventional formalism.

We have analyzed the question of uniqueness of PPS ensembles and the relation of uniqueness to the joint determination of noncommuting observables. We have argued that there are cases in which some noncommutative strong intermediate measurements might not change the PPS ensemble, while their outcomes can be ascertained.

The commutativity of weak measurements was explained as resulting from the usual approach toward approximations in physics, and was shown to possess no contradiction with the Heisenberg uncertainty principle.

The physical meaning of the weak values is centered around the claim that the effect of any interaction which is weak enough will depend on such weak values. We argued that the weak values can be regarded as evidence of the simultaneous existence of both boundary states in the intermediate period.

APPENDIX A: MEASUREMENT DESCRIPTION IN QUANTUM MECHANICS

In quantum mechanical description of a measurement, we consider the evolution of a composite system: the measuring device and the measured system. Let $|MD_i\rangle$ be the state of the measuring device and $|\psi_i\rangle$ be the state of the system before the measurement. Then the initial state of the composite system is $|\psi_i\rangle\,|MD_i\rangle$. The process of measurement is an interaction between the measuring device and the system. In the case of a linear interaction, the interaction Hamiltonian is $H_{\text{in}}(t) = -g(t)\,qA$, where A is the measured observable of the system and q is the canonical variable of the measuring device coupled to A. The quantity π, the conjugate momentum of q, will change in time according to Hamilton's equation:

$$\frac{d\pi}{dt} = -\frac{\partial H}{\partial q} = g(t)A \tag{A1}$$

The change in π due to the interaction is proportional to the value of A. Thus we can read the result of the measurement in the π-scale of the measuring device:

$$A \propto \delta\pi = \pi_f - \pi_i \tag{A2}$$

For a simple description of the evolution of the composite system during the measurement, we assume a zero free Hamiltonian (the result can be generalized for a free Hamiltonian which commutes with the interaction Hamiltonian). The evolution in time of the composite system is given by

$$\begin{aligned} |\psi(T)\rangle\, |MD(T)\rangle &= \exp\left[-i\int_0^T H_{\text{in}}(t)\, dt\right] |\psi_i\rangle\, |MD_i\rangle \\ &= \exp(iGqA)\, |\psi_i\rangle\, |MD_i\rangle \end{aligned} \tag{A3}$$

where $G = \int_0^T g(t)\, dt$. For a discrete observable $A\,|a_k\rangle = a_k\,|a_k\rangle$, we can write $|\psi_i\rangle = \sum_k c_k\,|a_k\rangle$; then

$$|\Psi(T)\rangle\, |MD(T)\rangle = \sum_k \exp(iGa_k)c_k\, |a_k\rangle\, |MD_i\rangle \tag{A4}$$

The term $\exp(iGqa_k)$ is a translation operator of $|MD_i\rangle$ in the π-representation

$$\exp(iGqa_k)\, MD(\pi) = MD(\pi - GA_k) \tag{A5}$$

Assuming $MD_i(q)$ to be a Gaussian, the state of the measuring device in the π-representation after the measurement is a superposition of Gaussians, each centered around a value corresponding to an eigenstate of the system. In strong measurements the Gaussians are sharp, so that their spread $\Delta\pi$ is small compared to the separations Δa_n. The collapse to an eigenstate of the system leaves the measuring device in the state corresponding to one of the Gaussians

$$\langle a_n|\, \exp(iqGa_n)\, |a_n\rangle\, |MD_i\rangle \tag{A6}$$

Thus, the total effect of the measurement on the measuring device is the shift Ga_n in the π-representation of the state of the measuring device.

APPENDIX B. PROOF OF THE CONSERVATION OF THE PROBABILITY TRANSITION OF A PPS ENSEMBLE UNDER SPECIFIC INTERVENING MEASUREMENTS

Aharonov and Vaidman[4] have proved that when the weak value of A equals one of the eigenvalues of A, then the result of a strong intervening measurement of A can be ascertained with probability unity. Here we show that in this case it also holds that the probability transition from the pre- to the post-selection state is undisturbed by adding this intervening strong measurement of A.

Case a: Simple States

It is given that

$$A_w = \frac{\langle\psi_2|\, A\, |\psi_1\rangle}{\langle\psi_2|\psi_1\rangle} = a_n \tag{B1}$$

$$A = \sum_k |a_k\rangle\, a_k \langle a_k| \tag{B2}$$

The probability amplitude normalized within the PPS is

$$\frac{\langle\psi_2|a_k\rangle\langle a_k|\psi_1\rangle}{\langle\psi_2|\psi_1\rangle}$$

Since a_n is measured with certainty, it also holds that

$$\langle\psi_2|a_k\rangle\langle a_k|\psi_1\rangle = 0, \qquad \text{for all} \quad k \neq n \tag{B3}$$

Thus

$$\langle\psi_2|A\,|\psi_1\rangle = a_n\langle\psi_2|a_n\rangle\langle a_n|\psi_1\rangle \tag{B4}$$

From Eqs. (A1) and (A4) we get

$$\langle\psi_2|a_n\rangle\langle a_n|\psi_1\rangle = \langle\psi_2|\psi_1\rangle \tag{B5}$$

which means that the transition probability amplitude is unchanged.

Case b: Generalized States

The generalized state

$$\sum \alpha_i\langle\psi_i|\ |\phi_i\rangle \tag{B6}$$

can be realized by making a measurement on a system $|\phi\rangle$ that is part of a composite system that includes the system $|\phi\rangle$ plus an external system $|\ \rangle_{ex}$. The composite system is pre- and post-selected for the states[4]

$$|\Psi_1\rangle = \sum_{i=1}^{N} \beta_i\, |i\rangle_{\text{ex}}\, |\phi_i\rangle$$
$$|\Psi_2\rangle = \sum_{i=1}^{N} \gamma_i\, |i\rangle_{\text{ex}}\, |\psi_i\rangle \tag{B7}$$

with α_i given by

$$\alpha_i = \beta_i \gamma_i^*$$

Without loss of generalization we can choose

$$|\Psi_1\rangle = \sum_{i=1}^{N} \alpha_i\, |i\rangle_{\text{ex}}\, |\phi_i\rangle$$
$$|\Psi_2\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle_{\text{ex}}\, |\psi_i\rangle \tag{B8}$$

The intervening measurement on the partial system only is

$$A' = A \otimes 1_{\text{ex}} \tag{B9}$$

For this case the weak value is

$$A'_w = A_w = \frac{\langle \Psi_2|\, A\, |\Psi_1\rangle}{\langle \Psi_2 | \Psi_1\rangle} = \frac{1/\sqrt{N} \sum_{i=1}^{N} \langle i|_{\text{ex}} \langle \psi_i|\, A \sum_{k=1}^{N} \alpha_k\, |k\rangle_{\text{ex}}\, |\phi_k\rangle}{1/\sqrt{N} \sum_{i=1}^{N} \langle i|_{\text{ex}} \langle \psi_i| \sum_{k=1}^{N} \alpha_k\, |k\rangle_{\text{ex}}\, |\phi_k\rangle}$$
$$= \frac{1/\sqrt{N} \sum_{i=1}^{N} \alpha_i \langle \psi_i|\, A\, |\phi_i\rangle}{1/\sqrt{N} \sum_{i=1}^{N} \alpha_i \langle \psi_i | \phi_i\rangle} = a_n \tag{B10}$$

The condition that a_n is measured with certainty is given by

$$\sum_{i=1}^{N} \alpha_i \langle \psi_i | \alpha_k\rangle \langle a_k | \phi_i\rangle = 0, \qquad \text{for all} \quad k \neq n \tag{B11}$$

By analogy to the previous derivation, we get

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \alpha_i \langle \psi_i | a_n\rangle \langle a_n | \phi_i\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \alpha_i \langle \psi_i | \phi_i\rangle = \langle \Psi_2 | \Psi_1\rangle \tag{B12}$$

which leads to the same conclusion as in case a.

APPENDIX C: AN EXAMPLE OF THE INSIGNIFICANT EFFECT OF THE WEAK MEASUREMENT ON THE SYSTEM

In the following example we use standard Von Neumann measurement theory to show that the effect of a weak measurement on a system is negligible. This example also illustrates how a strong measurement on a pure ensemble of quantum systems can result in identical weak measurements for each system.

We assume the initial state of N systems to be

$$|\psi_i\rangle = \prod_{k=1}^{N} |\sigma_z^k = +1\rangle \tag{C1}$$

For simplicity, we assume the intial state of the measuring device in the π-representation to be a Gaussian with a small spread Δ (all formulas are given up to normalization constants):

$$\langle \pi | MD_i \rangle = \exp(-\pi^2/2\Delta^2) \tag{C2}$$

and in the q-representation

$$\langle q | MD_i \rangle = \Delta \exp[-(\Delta q)^2/2] \tag{C3}$$

Suppose a strong measurement of $J_x = \sum_{k=1}^{N} (1/N)\sigma_x^k$ was performed and found to be equal to 1, its maximum possible value. Thus the known final state of the measuring device is

$$\begin{aligned} \langle \pi | MD_f \rangle &= \exp[-(\pi - 1)^2/2\Delta^2] \\ \langle q | MD_f \rangle &= \Delta \exp(iq) \exp[-(\Delta q)^2/2] \end{aligned} \tag{C4}$$

The interaction Hamiltonian is

$$H_{\text{in}}(t) = -g(t) q \sum_{k=1}^{N} \frac{1}{N} \sigma_x^k \tag{C5}$$

and we assume that

$$\int_0^T g(t)\, dt = 1$$

After the measurement is over, the final state of the ensemble is

$$\begin{aligned} |\psi_f\rangle &= \langle MD_f| \exp\left[-i \int_0^T H_{\text{in}}(t)\, dt\right] |MD_i\rangle\, |\psi_i\rangle \\ &= \Delta^2 \int_{-\infty}^{\infty} \exp[-(\Delta q)^2/2] \exp(-iq) \exp[-(\Delta q)^2/2] \\ &\quad \times \exp\left(iq \sum_{k=1}^{N} \frac{1}{N} \sigma_x^k\right) \prod_{k=1}^{N} |\sigma_z^k = +1\rangle\, dq \end{aligned}$$

$$= \Delta^2 \int_{-\infty}^{\infty} \exp[-(\Delta q)^2 - iq] \times \prod_{k=1}^{N} \left[\cos(q/N)\, |\sigma_z^k = +1\rangle + i \sin(q/N)\, |\sigma_z^k = -1\rangle \right] dq$$

$$\approx \Delta^2 \int_{-\infty}^{\infty} \exp\left[-(\Delta q)^2 - iq \right] \prod_{k=1}^{N} \left[|\sigma_z^k = +1\rangle + (iq/N)\, |\sigma_z^k = -1\rangle \right] dq \tag{C6}$$

For large N the final state is insignificantly different from the state in which all spins are $|\sigma_z = +1\rangle$, which was the initial state.

ACKNOWLEDGMENTS

We are very grateful to Yakir Aharonov and Lev Vaidman for introducing us to their work and for their stimulating remarks. Special thanks are due to Larry P. Horwitz.

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