Quantum Measurement of a Single System

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Yoshihisa Yamamoto

With new technologies permitting the observation and manipulation of single quantum systems, the quantum theory of measurement is fast becoming a subject of experimental investigation in laboratories worldwide. This original new work addresses open fundamental questions in quantum mechanics in light of these experimental developments.

Using a novel analytical approach developed by the authors, *Quantum Measurement of a Single System* provides answers to three long-standing questions that have been debated by such thinkers as Bohr, Einstein, Heisenberg, and Schrödinger. It establishes the quantum theoretical limits to information obtained in the measurement of a single system on the quantum wavefunction of the system, the time evolution of the quantum observables associated with the system, and the classical potentials or forces which shape this time evolution. The technological relevance of the theory is also demonstrated through examples from atomic physics, quantum optics, and mesoscopic physics.

Suitable for professionals, students, or readers with a general interest in quantum mechanics, the book features recent formulations as well as humorous illustrations of the basic concepts of quantum measurement. Researchers in physics and engineering will find *Quantum Measurement of a Single System* a timely guide to one of the most stimulating fields of science today.

ORLY ALTER, PhD, is currently a postdoctoral fellow in the Department of Genetics at Stanford University. **YOSHIHISA YAMAMOTO, PhD,** is a professor in the Departments of Applied Physics and Electrical Engineering at Stanford University. He is currently the director of the ICORP Quantum Entanglement Project of the Japanese Science and Technology (JST) Corporation. While they collaborated on the research presented in this book, Yamamoto was the director of the ERATO Quantum Fluctuation Project of JST, and Alter was a doctoral student at the Department of Applied Physics at Stanford. She was selected as a finalist for the American Physical Society Award for Outstanding Doctoral Thesis Research in Atomic, Molecular or Optical Physics for 1998 for this work.

Cover Illustration: David B. Oberman

ALTER YAMAMOTO

Quantum Measurement of a Single System
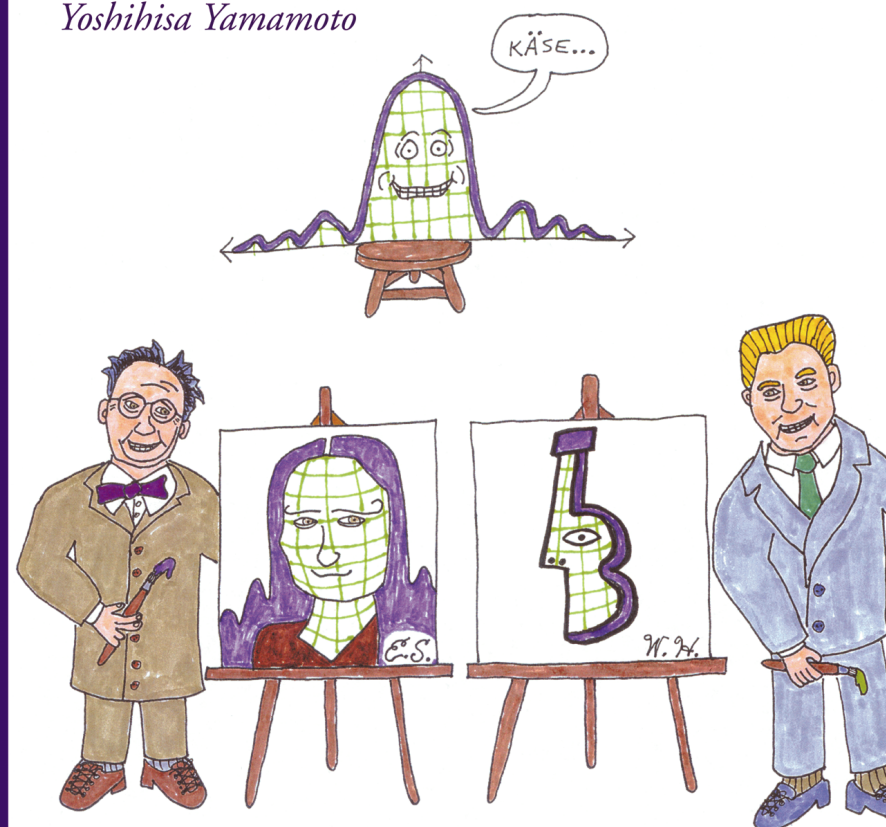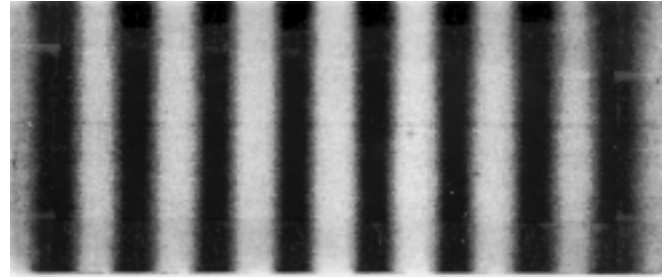
Motivation (I)

Technological advances allow control of single quantum systems:
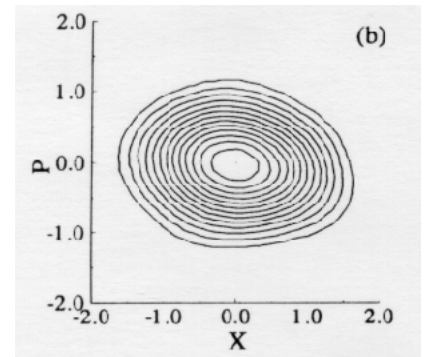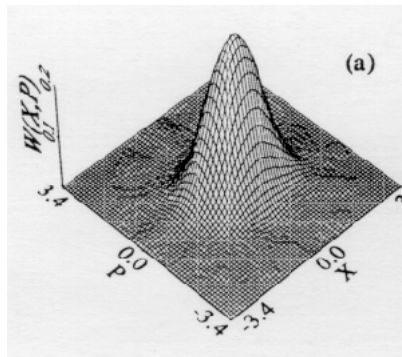
Squeezing of single wavepackets of light
Trapping of single atoms, ions or DNA molecules

What is the meaning
of the quantum state
(or wavefunction)?

Smithey, Beck,
Raymer & Faridani,
PRL 70, 1244 (1993).

Vogel & Risken,
PRA 40, 2847 (1989).

Aharonov, Anandan & Vaidman, *PRA* 47, 4616 (1993):

It may be possible to determine the unknown quantum wavefunction of a single system, and give the wavefunction a physical meaning, in addition to its statistical meaning.

Imamoglu, *PRA Rapid Communication* 47, R4577 (1993);
Royer, *PRL* 73, 913 (1994) [Erratum in *PRL* 74, 1040 (1995)].
Huttner, private communication (1995).

Motivation (II)

State-of-the-art precision measurements are based on monitoring the time evolution of a single physical system:

<div align="center">

Gravitational wave detection
Scanning microscopy (AFMs)
Josephson junction circuits (SQUIDs)

</div>

What is the fundamental limit to the determination of the time evolution of a single system?

What is the fundamental quantum limit to the detection of a classical signal via the monitoring of a single system?

<div align="center">

Hollenhorst, *PRD* 19, 1669 (1979);
Braginsky, Vorontsov & Thorne, *Science* 209, 541 (1980);
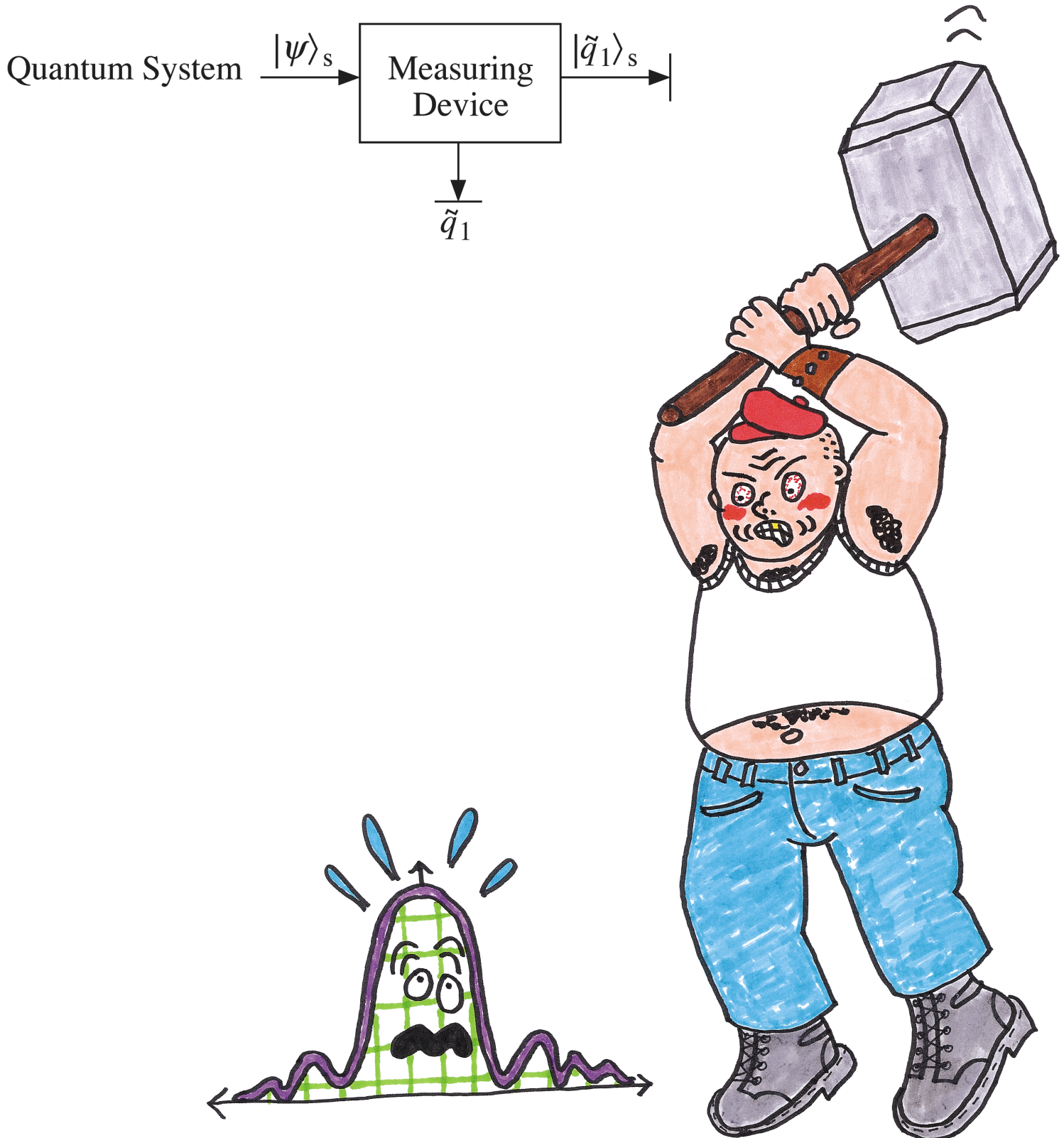Caves, Thorne, Drever, Sandberg & Zimmermann, *RMP* 52, 341 (1980);
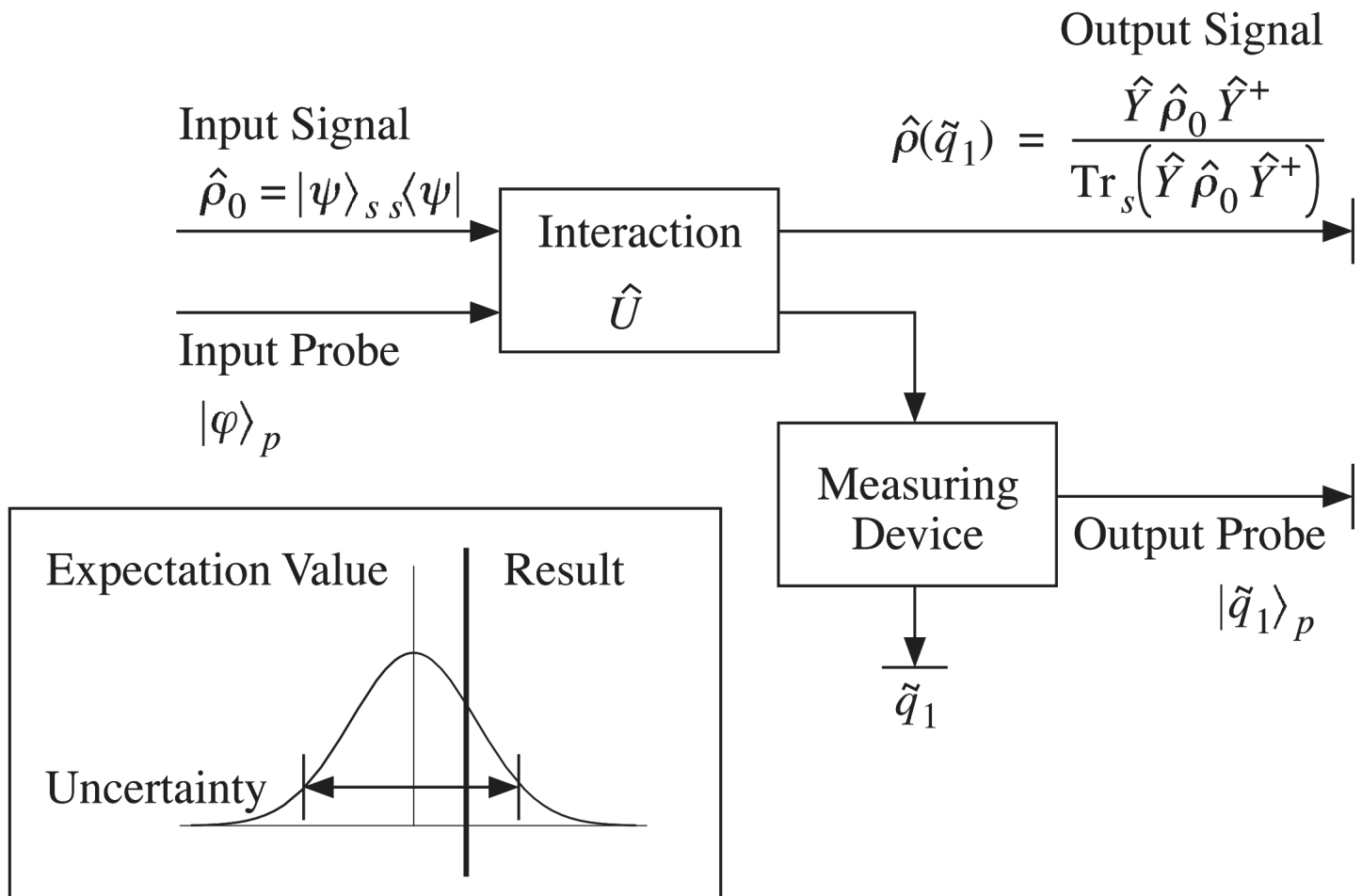Yuen, *PRL* 51, 719 (1983):

There may be no such fundamental limit.

</div>

The Projection Postulate

Quantum System $\xrightarrow{\quad |\psi\rangle_s \quad}$ | Measuring Device | $\xrightarrow{\quad |\tilde{q}_1\rangle_s \quad}$

$\downarrow$

$\tilde{q}_1$

Generalized Quantum Measurement

Input Signal
$$\hat{\rho}_0 = |\psi\rangle_{s\,s}\langle\psi|$$

Input Probe
$$|\varphi\rangle_p$$

Interaction $\hat{U}$

Output Signal
$$\hat{\rho}(\tilde{q}_1) = \frac{\hat{Y}\,\hat{\rho}_0\,\hat{Y}^+}{\mathrm{Tr}_s\!\left(\hat{Y}\,\hat{\rho}_0\,\hat{Y}^+\right)}$$

Measuring Device

Output Probe
$$|\tilde{q}_1\rangle_p$$

$$\tilde{q}_1$$

Expectation Value — Result

Uncertainty

The generalized quantum measurement is described by the generalized projection operator

$$\hat{Y}(\hat{q},\tilde{q}_1) = {}_p\langle\tilde{q}_1|\hat{U}|\phi\rangle_p \ .$$

Unitary signal-probe interaction $\hat{U} \rightarrow$ deterministic change in $\hat{\rho}_0$.

Projection in the measurement of the probe $\hat{Y} \rightarrow$ reduction, i.e., stochastic change in $\hat{\rho}_0$.

Outline

# Impossibility of Determining the Quantum Wavefunction of a Single System

Model: A series of QND measurements of the photon-number $\hat{n}$ is performed on a single wavepacket of light, which is initially in the pure state $\hat{\rho}_0$.

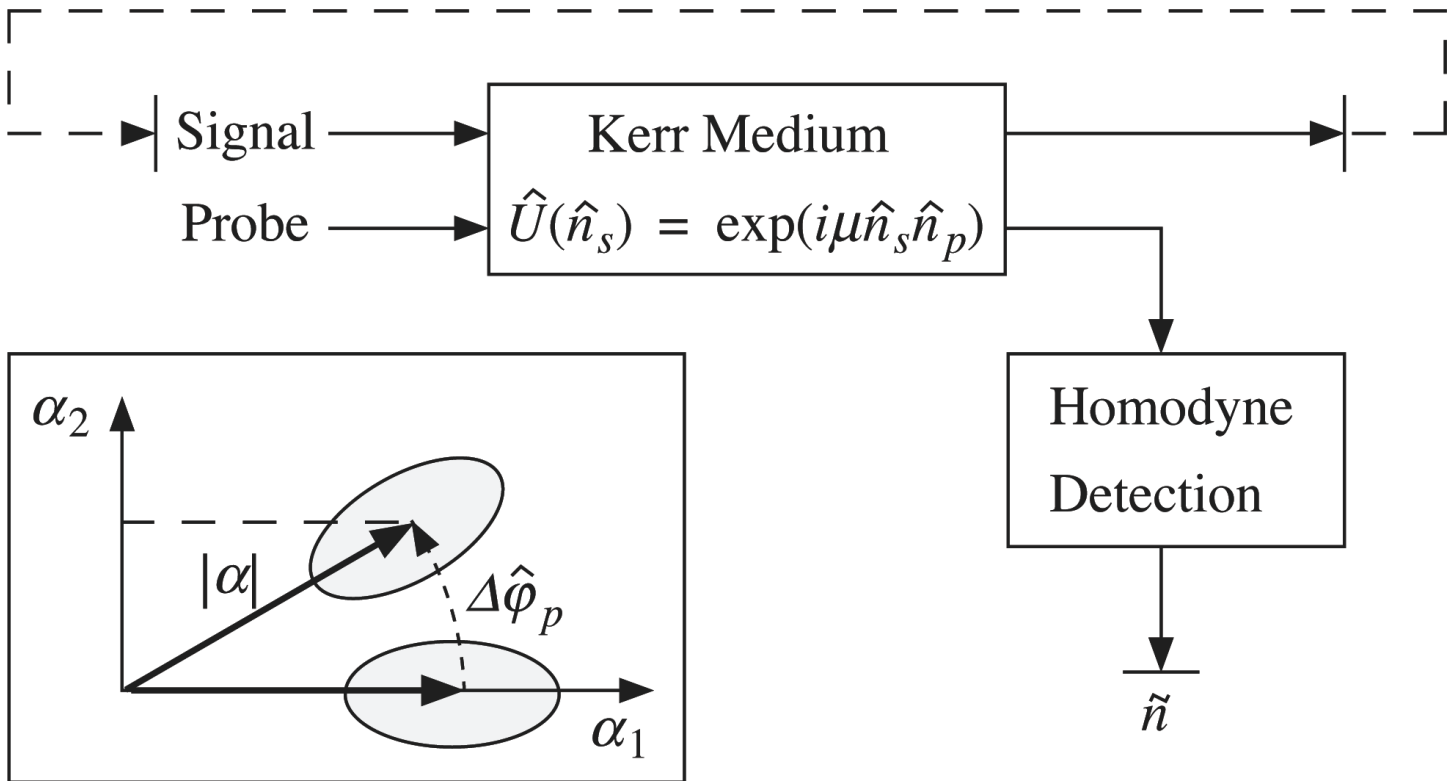**Goal:** Use the statistics of the measurement results to estimate the initial probability density,

$$P_0(n) \ = \ \langle n|\hat{\rho}_0|n\rangle$$

initial expectation value $\qquad \langle n_0 \rangle$

and initial uncertainty $\qquad \langle \Delta n_0^2 \rangle$.
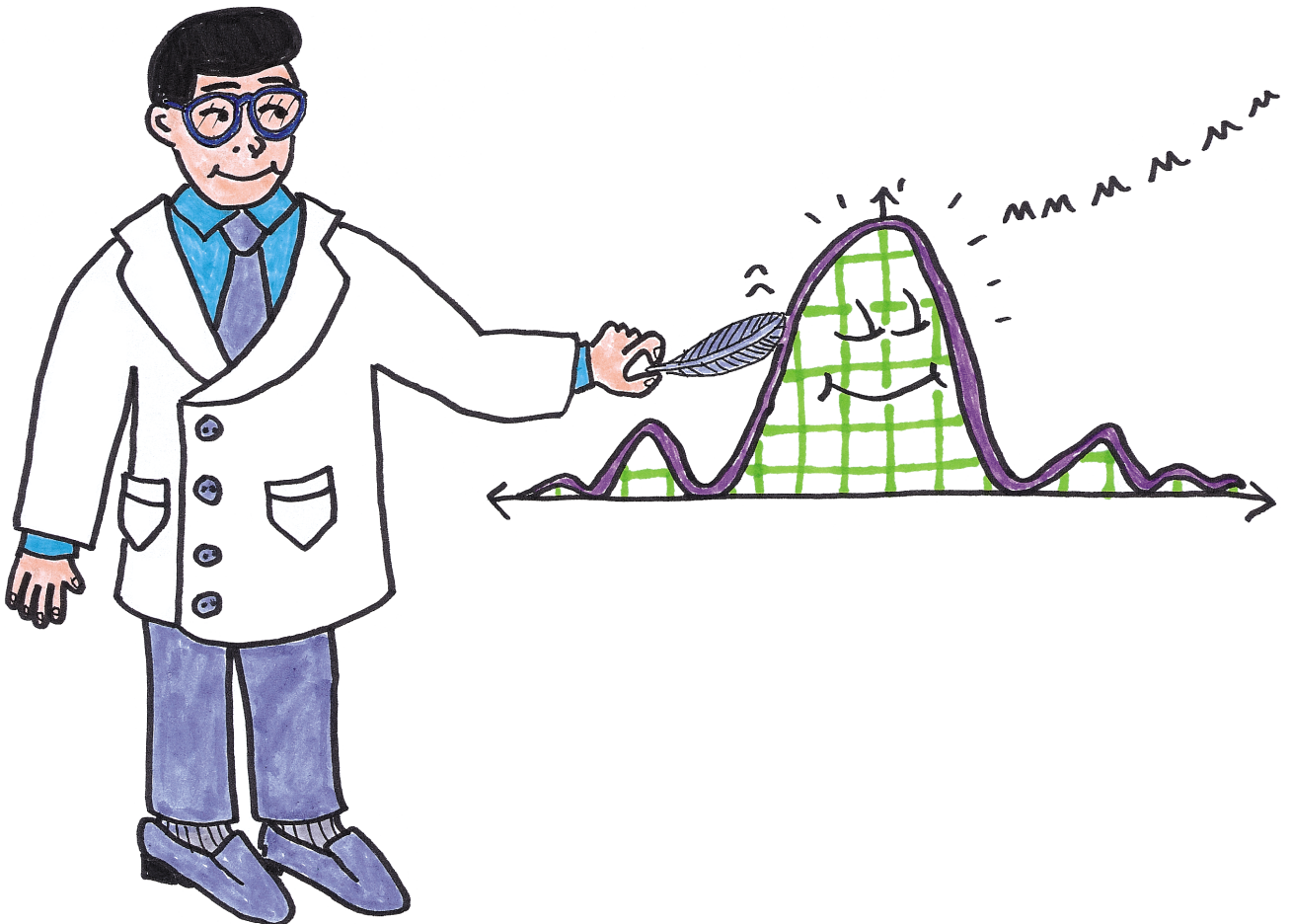
A Series of QND Measurements of a Single System

In each measurement, the signal photon-number is estimated from the change in the probe phase, which equals the second quadrature-amplitude of the probe approximately

with the estimation error $\qquad\qquad \langle \Delta \tilde{n}^2 \rangle = \langle \Delta n_0^2 \rangle + \Delta_m^2$

Saturated Quantum Brownian Motion and Continuous Wavefunction Collapse

Photon-Number

n_0

Δ_0

$10^0 \quad 10^1 \quad 10^2 \quad 10^3 \quad 10^4 \quad 10^5$

Number of Measurements
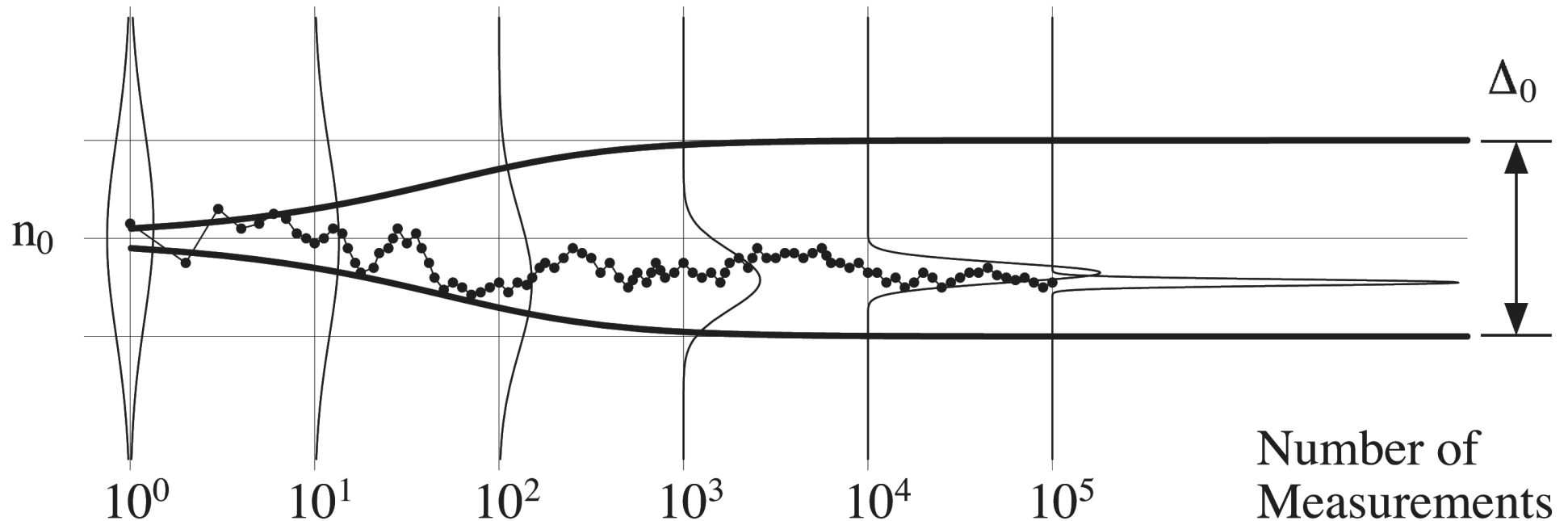
In terms of the changes induced in the wavefunction, a series of imprecise measurements is equivalent to a single precise measurement.

Statistics of the Measurement Results

NEW

Each measurement result, $\tilde{n}_1$ or $\tilde{n}_2$, estimates the initial expectation value

$$\langle \tilde{n}_1 \rangle = \langle n_0 \rangle$$

$$\langle \tilde{n}_2 \rangle = \langle n_0 \rangle$$

and second moment

$$\langle \tilde{n}_1^2 \rangle = \langle n_0^2 \rangle + \Delta_m^2$$

$$\langle \tilde{n}_2^2 \rangle = \langle n_0^2 \rangle + \Delta_m^2$$

but not the initial uncertainty

$$\langle \Delta n_0^2 \rangle = \langle n_0^2 \rangle - \langle n_0 \rangle^2$$

If $\tilde{n}_1$ and $\tilde{n}_2$ were independent results, obtained from two different quantum systems, then

$$\langle \tilde{n}_1 \tilde{n}_2 \rangle = \langle n_0 \rangle^2$$

and the initial uncertainty can be estimated.

In our case $\tilde{n}_2$ depends on $\tilde{n}_1$,

$$\langle \tilde{n}_1 \tilde{n}_2 \rangle = \langle n_0^2 \rangle$$

and the initial uncertainty cannot be estimated.

Conclusions

The unknown wavefunction of a single system cannot be determined from the results of a series of quantum measurements, due to the reduction which is induced by the measurement process.

The quantum wavefunction has only an epistemological meaning:
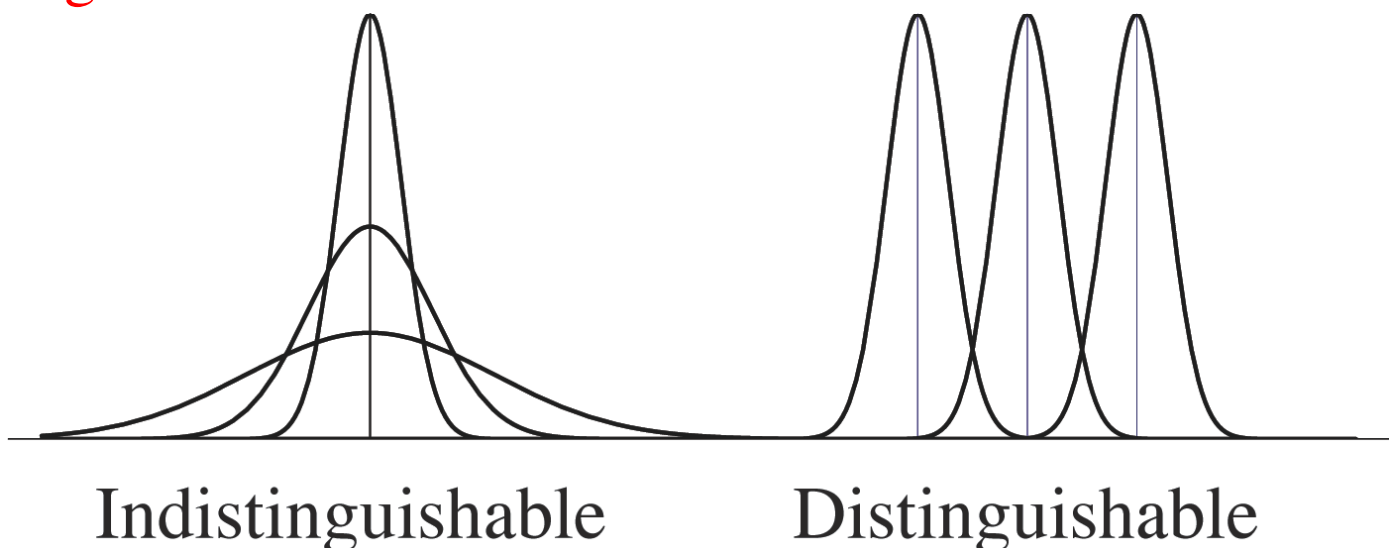
→ The quantum uncertainty

$$\langle \Delta q_0^2 \rangle = \langle q_0^2 \rangle - \langle q_0 \rangle^2$$
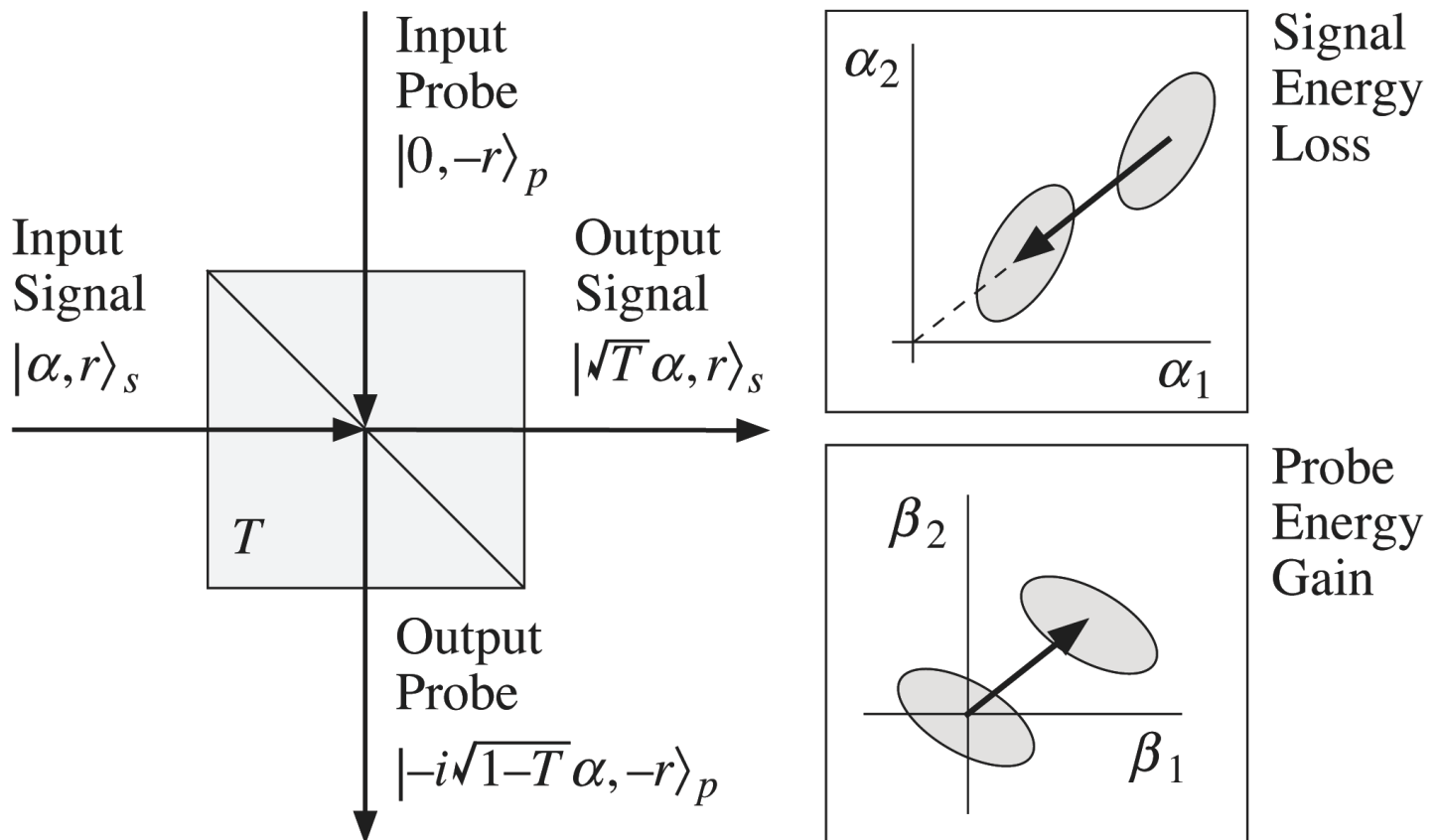
is not an observable.
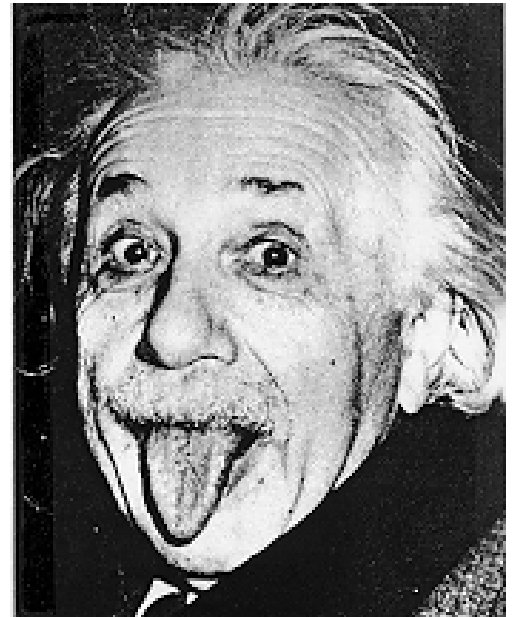
→ Quantum mechanics is not an ergodic theory.

→ Information in quantum communication channels cannot be coded on the uncertainties of the quantum signals.

Indistinguishable   Distinguishable

Measurements without Entanglement of a Squeezed Wavepacket of Light

Input Probe $|0,-r\rangle_p$

Input Signal $|\alpha,r\rangle_s$

T

Output Signal $|\sqrt{T}\,\alpha,r\rangle_s$

Output Probe $|-i\sqrt{1-T}\,\alpha,-r\rangle_p$

α_2

α_1

Signal Energy Loss

β_2

β_1

Probe Energy Gain

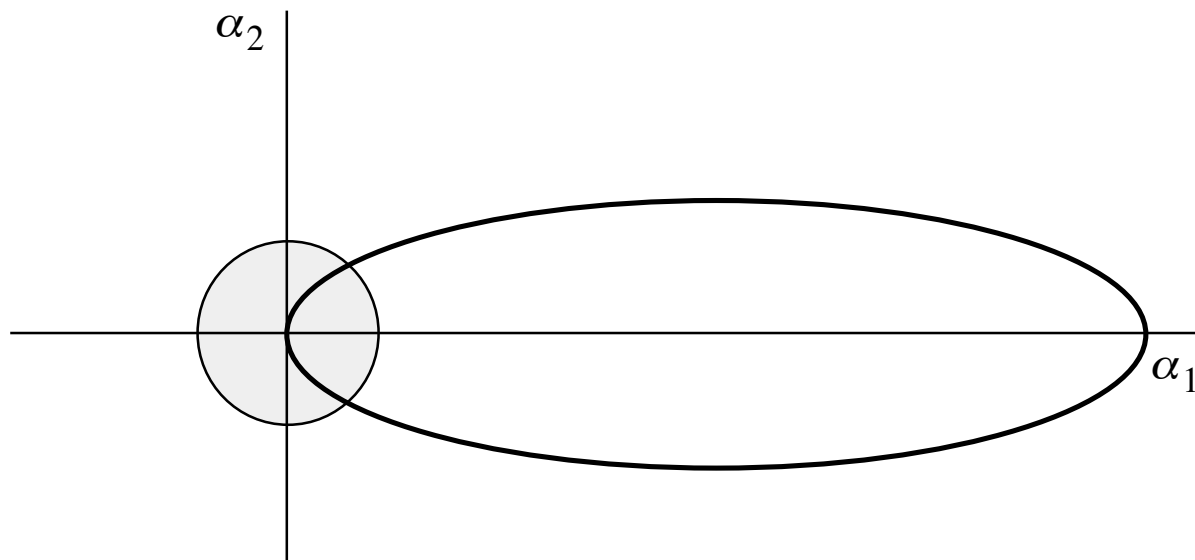# Conclusions

Measurements without entanglement avoid the reduction and induce only a deterministic change in the quantum state of the measured system by utilizing some a-priori information about this state.

This is the only additional information, which is present in the statistics of the results of a series of measurements of the single system.

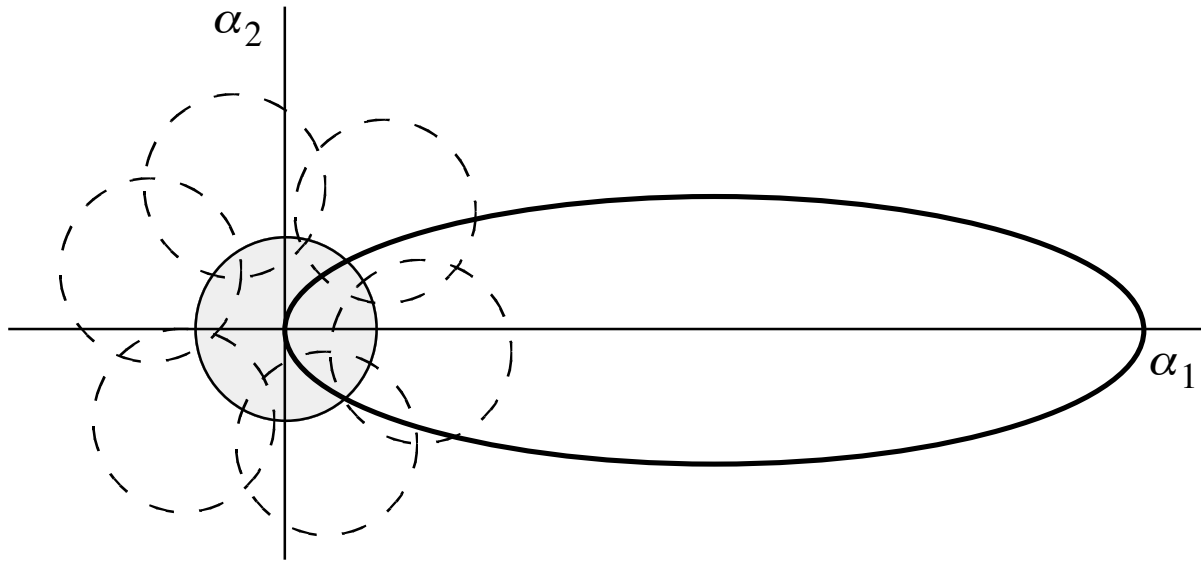# Adiabatic Position Measurement of a Harmonic Oscillator

$$\hat{U}(T)\,|0\rangle_s\,|\beta_1\rangle_p$$

$$\approx\ |0\rangle_s\,|\beta_1\rangle_p$$
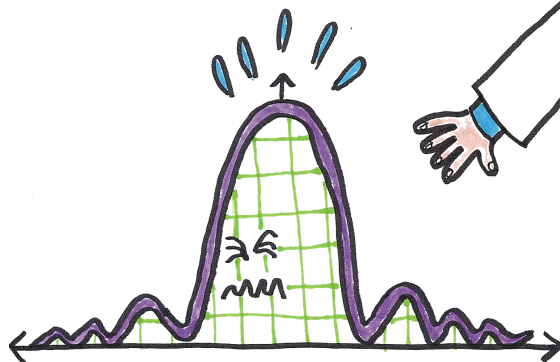
Adiabatic Position Measurement of a Harmonic Oscillator

$$\hat{U}(T)\,|0\rangle_s\,|\beta_1\rangle_p \;=\; e^{i\phi(\beta_1,T)}\,\left|e^{-i\omega T}\,\delta(\beta_1,T)\right\rangle_s\,|\beta_1\rangle_p$$
$$\approx\; |0\rangle_s\,|\beta_1\rangle_p$$

Conclusions:

An adiabatic interaction leaves the signal and the probe only approximately disentangled. The signal is not protected from reduction.

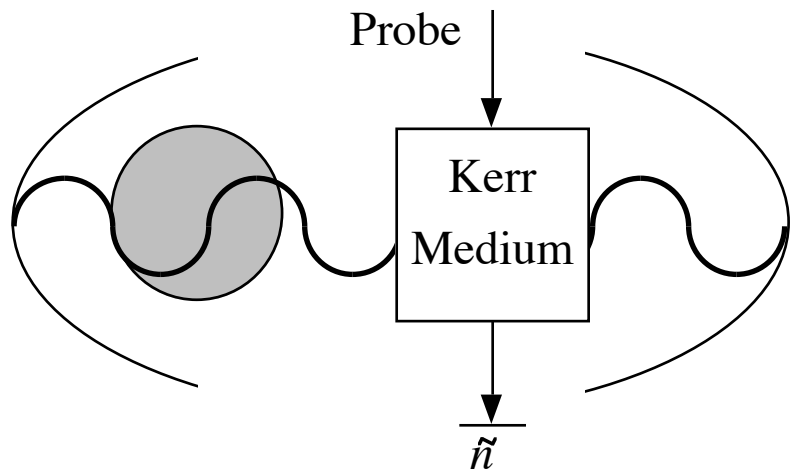# Limit to Monitoring the Time Evolution of a Single System

Model: A series of QND measurements of the photon-number $\hat{n}$ of a two-level atom in a single-photon mode cavity during its time evolution.

Probe

Kerr Medium

Estimation Error

$$\langle \Delta \tilde{n}^2 \rangle = \langle \Delta n_0^2 \rangle + \Delta_m^2$$

$\tilde{n}$

Goal: Use the measurement results to estimate the Rabi oscillations of the energy in the cavity.

Cavity Photon-Number
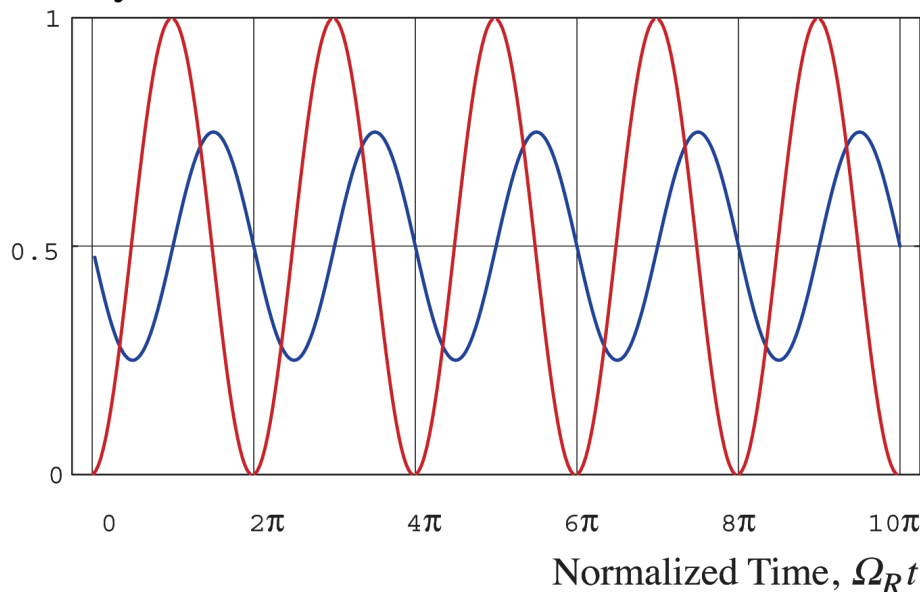
Initial State

$$|\psi(0)\rangle = \frac{e^{-i\pi/12}}{\sqrt{2}} |e\rangle_a |0\rangle_p$$
$$\pm \frac{e^{i\pi/12}}{\sqrt{2}} |g\rangle_a |1\rangle_p$$

$$|\psi(0)\rangle = |e\rangle_a |0\rangle_p$$

Normalized Time, $\Omega_R t$

Quantum Zeno Effect of a Single System

Imprecise Measurements $\Delta_m = 2$

(a) Ensemble Averaged Measured Photon-Number

(b) Photon-Number Measured on a Single System

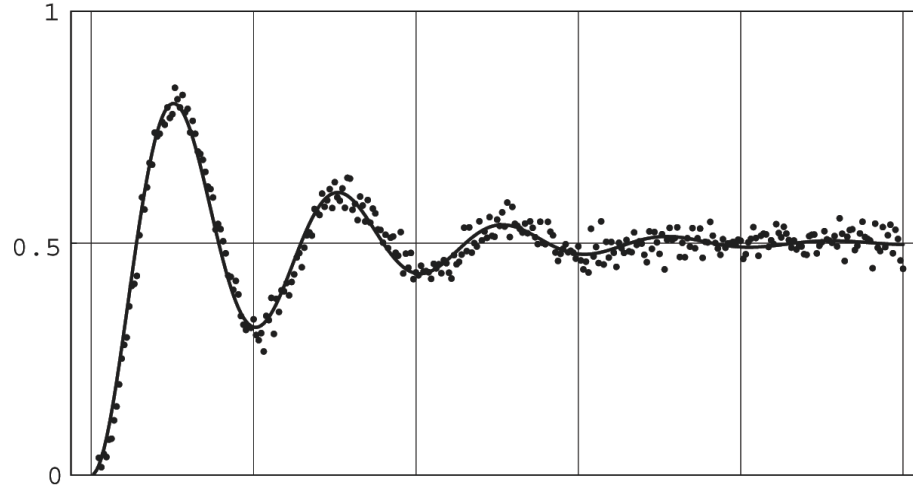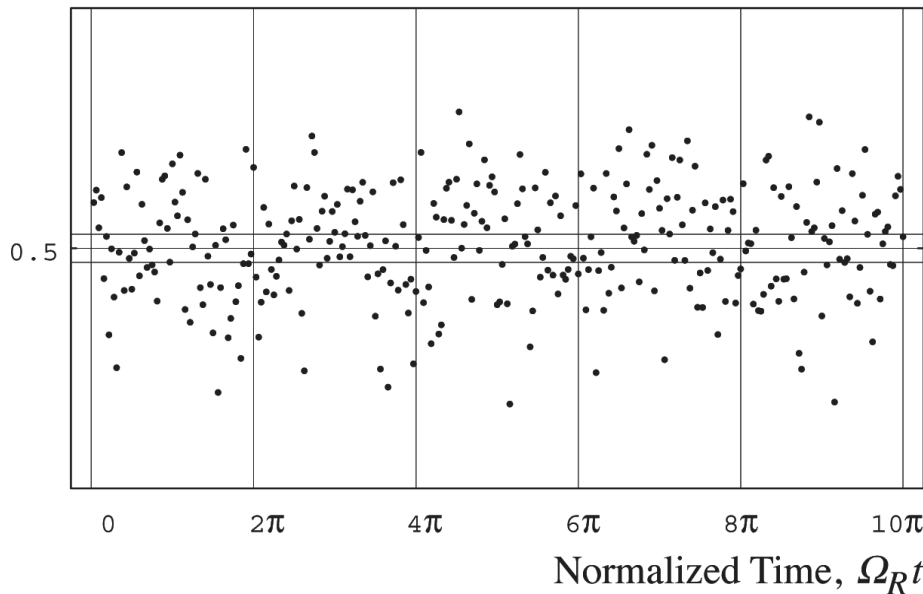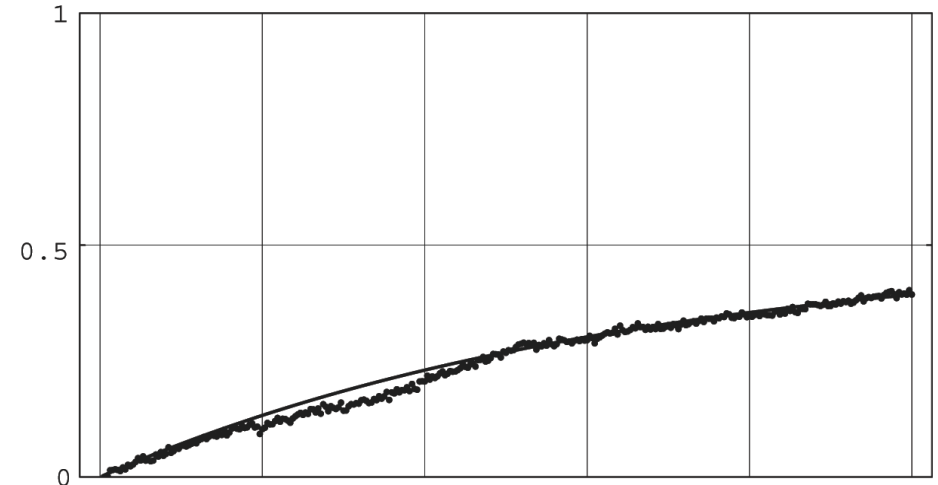Precise Measurements $\Delta_m = 0.1$

(a) Ensemble Averaged Measured Photon-Number

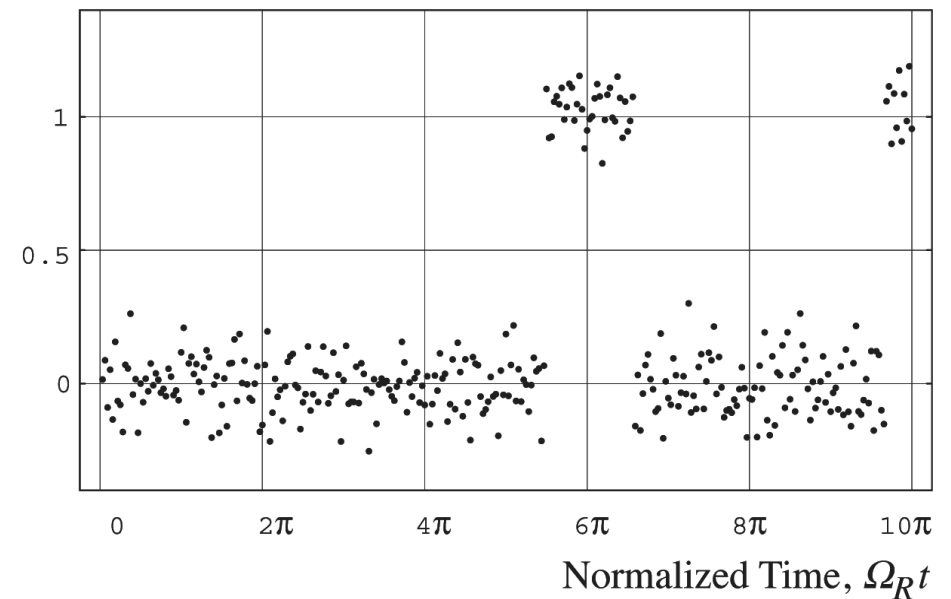(b) Photon-Number Measured on a Single System

Normalized Time, $\Omega_R t$

Ensemble $\rightarrow$ unitary time evolution $\rightarrow$ initial quantum state;

Single system $\rightarrow$ no initial quantum state $\rightarrow$ no time evolution.

Schrödinger and Heisenberg Pictures

Quantum Zeno effect of a single system – Schrödinger picture: a series of n measurements of the observable $\hat{q}$ of a single system during its time evolution.
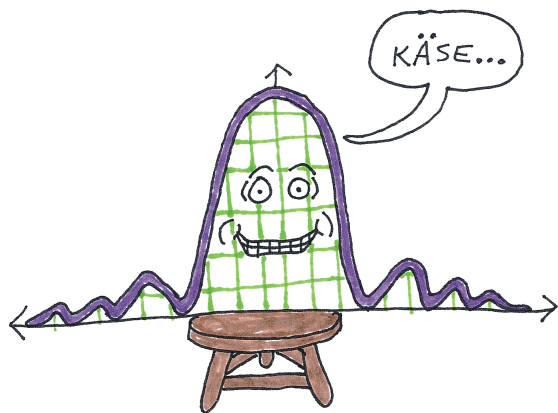
$$P_S(\tilde{q}_1, \ldots, \tilde{q}_n) =$$

$$\mathrm{Tr}_s[\hat{Y}_n \hat{U}_n \ldots \hat{Y}_1 \hat{U}_1 \hat{\rho}_0 \hat{U}_1^+ \hat{Y}_1^+ \ldots \hat{U}_n^+ \hat{Y}_n^+] =$$

$$\mathrm{Tr}_s[\hat{Y}(\hat{q}_n,\tilde{q}_n) \ldots \hat{Y}(\hat{q}_1,\tilde{q}_1) \, \hat{\rho}_0 \, \hat{Y}^+(\hat{q}_1,\tilde{q}_1) \ldots \hat{Y}^+(\hat{q}_n,\tilde{q}_n)] =$$

$$P_H(\tilde{q}_1, \ldots, \tilde{q}_n)$$

Impossibility of determining the quantum wavefunction of a single system – Heisenberg picture: a series of n measurements of time-varying observables of a single system, with no time evolution between successive measurements.

Conclusions

The quantum Zeno effect of a single system and the impossibility of determining the wavefunction of a single system are equivalent.

In the Heisenberg picture, the series of measurement results cannot determine the initial quantum state of the s y s t e m, and in the Schrödinger picture, these results cannot determine the unitary time evolution of the single system.

The quantum Zeno effect is more than a dephasing effect: It is a quantum measurement effect.

The monitoring of the time evolution of a single system is limited by the impossibility of determining the quantum state of this system.

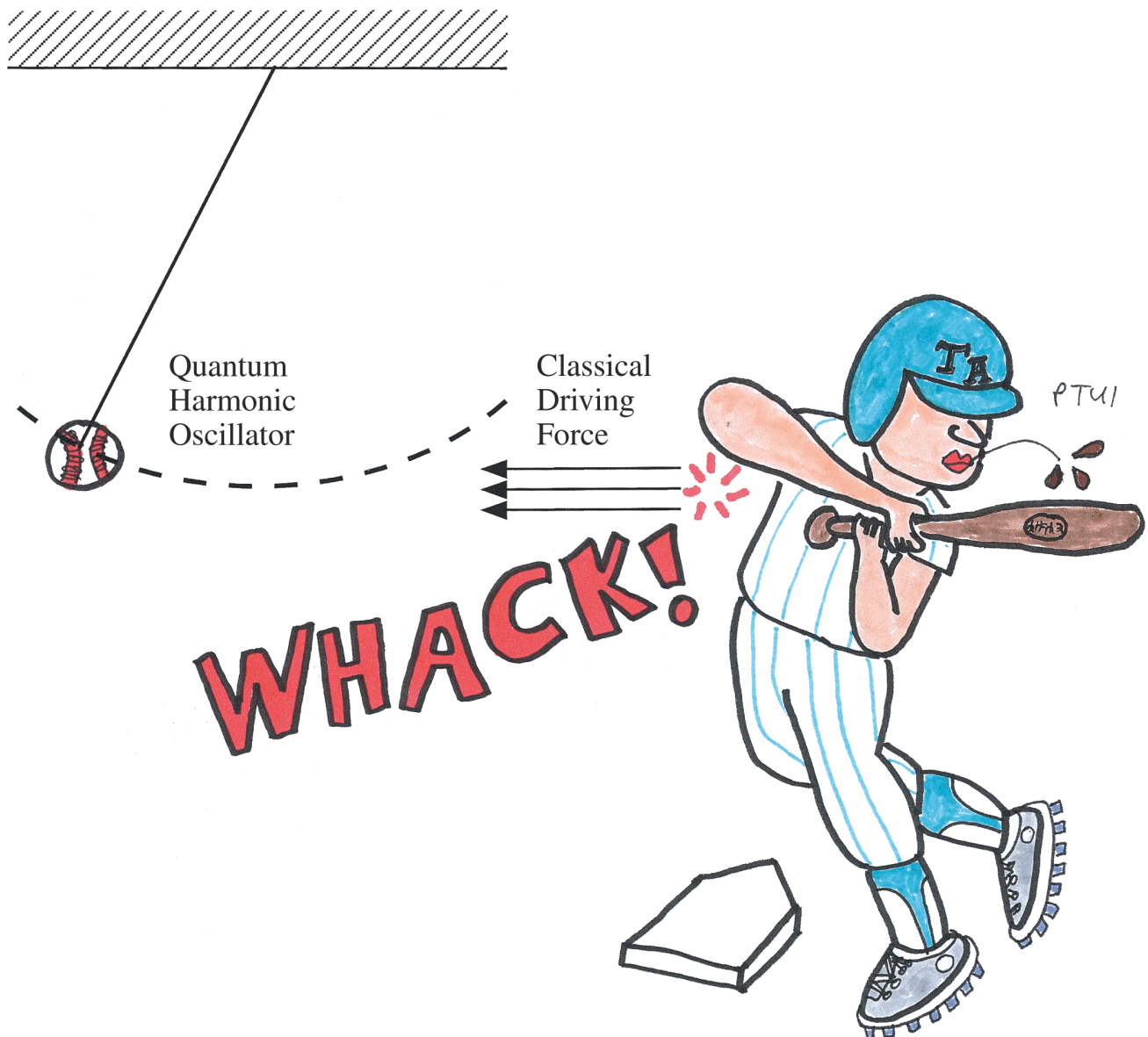# Limit to Precision Measurements with a Single Quantum System

Model: Measurements of a quantum harmonic oscillator (free mass) driven by a classical force:

Gravitational wave detection
Scanning microscopy (AFMs)
Josephson junction circuits (SQUIDs)

Goal: Use the measurement results to estimate the magnitude and phase of the force.

Monitoring the Momentum of a Driven Free Mass

$$\hat{p}(t) - \hat{p}(0) = \int_0^t dt'\, F(t')$$

Common assumption of independent errors in the estimates of $\hat{p}(0)$ and $\hat{p}(t)$:

$$\langle \Delta \hat{p}^2(t) \rangle + \Delta_m^2 = \langle \Delta \hat{p}^2(0) \rangle + \Delta_m^2$$

→ Force detection is best when $\langle \Delta \hat{p}^2(0) \rangle = 0$;

→ Interest in initial quantum state preparation.

However, force detection is limited by the error in the estimate of $\hat{p}(t) - \hat{p}(0)$ that is independent of $\langle \Delta \hat{p}^2(0) \rangle$:

$$\langle \Delta[\hat{p}(t) - \hat{p}(0)]^2 \rangle =$$
$$= \langle \Delta \hat{p}^2(t) \rangle + \langle \Delta \hat{p}^2(0) \rangle - \langle \{ \Delta \hat{p}(t), \Delta \hat{p}(0) \} \rangle$$
$$= \langle \Delta \hat{p}^2(t) \rangle - \langle \Delta \hat{p}^2(0) \rangle = 0$$

This is because of the correlation between $\hat{p}(0)$ and $\hat{p}(t)$, which is ignored by the common assumption.

→ Force detection is independent of the initial quantum state of the driven mass (or oscillator).

This agrees with the impossibility of estimating $\langle \Delta \hat{p}^2(0) \rangle$ of a single mass in an unknown state.
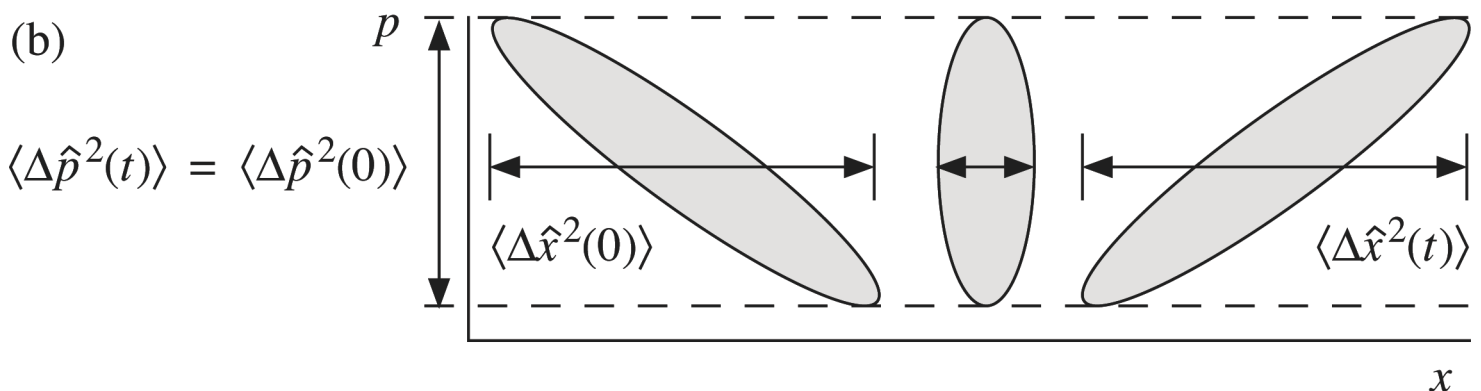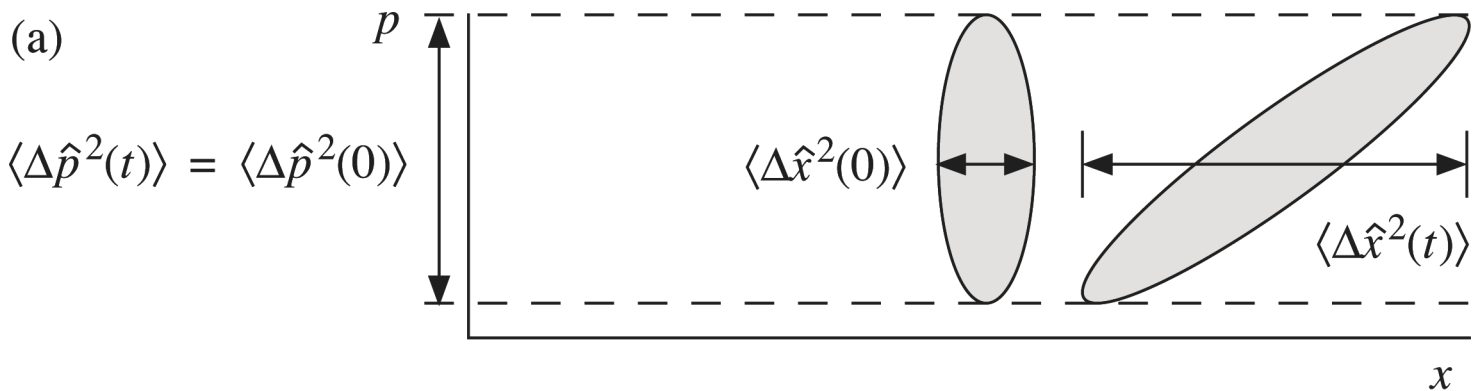
Standard Quantum Limit in Position Monitoring?

$$\hat{x}(t) - \hat{x}(0) = \hat{p}(0)\, t/m + \int_0^t dt' \int_0^{t'} dt'' \, F(t'')/m$$

Braginsky, *JETP* <u>26</u>, 831 (1968); Caves, Thorne, Drever, Sandberg & Zimmermann, *RMP* <u>52</u>, 341 (1980); Yuen, *PRL* <u>51</u>, 719 (1983).

Contractive state measurements overcome the standard quantum limit when the state of the mass is reset to a known contractive state with negative $\hat{x}(0)$ and $\hat{p}(0)$ correlation after each position measurement:

$$\langle \Delta \hat{x}^2(t) \rangle = \langle \Delta \hat{x}^2(0) \rangle + \langle \Delta \hat{p}^2(0) \rangle \, t^2/m^2$$

$$+ \langle \{ \Delta \hat{x}(0), \Delta \hat{p}(0) \} \rangle \, t/m \;\rightarrow\; 0$$

(a)

$\langle \Delta \hat{p}^2(t) \rangle = \langle \Delta \hat{p}^2(0) \rangle$

(b)

$\langle \Delta \hat{p}^2(t) \rangle = \langle \Delta \hat{p}^2(0) \rangle$

Standard Quantum Limit in Position Monitoring!

$$\hat{x}(t) - \hat{x}(0) = \hat{p}(0)\, t/m + \int_0^t dt' \int_0^{t'} dt''\, F(t'')/m$$

However, force detection is limited by the error in the estimate of the displacement that depends on $\langle \Delta\hat{p}^2(0)\rangle$

$$\langle \Delta[\hat{x}(t) - \hat{x}(0)]^2 \rangle =$$

$$= \langle \Delta\hat{x}^2(t)\rangle - \langle \Delta\hat{x}^2(0)\rangle - \langle \{\Delta\hat{x}(0), \Delta\hat{p}(0)\}\rangle\, t/m$$

$$= \langle \Delta\hat{p}^2(0)\rangle\, t^2/m^2 \;\rightarrow\; \infty$$

This is because of the correlation between $\hat{x}(0)$ and $\hat{x}(t)$ that was neglected in all previous analyses.

→ In exact position monitoring of a driven mass (or oscillator), all information about the driving force is lost.

→ From the trade-off between $\langle \Delta\hat{p}^2(0)\rangle$ and Δ_m^2 force detection is limited by $\langle \Delta[\hat{x}(t) - \hat{x}(0)]^2\rangle + \Delta_m^2 \geq \hbar t/m$ and that is, in fact, the standard quantum limit.

Caves & Milburn, *PRA* <u>36</u>, 5543 (1987); Mabuchi, *PRA* <u>58</u>, 123 (1998).

This agrees with the impossibility of determining both unknown $\hat{x}(0)$ and $\hat{p}(0)$ of a single mass (or oscillator).

Uncertainty Principle and Completeness of Quantum Theory

$$\langle \Delta \hat{x}^2(t) \rangle \langle \Delta \hat{p}^2(t) \rangle \geq \hbar^2 / 4$$

What are the limits to the information that can be obtained in the quantum measurement of a single system?

$$\hat{x}(t) - \hat{x}(0) = \hat{p}(0)\, t/m$$

Heisenberg, *Physical Principles of the Quantum Theory* (1930);
Schrödinger, *Interpretation of Quantum Mechanics* (1955);
Einstein, Tolman & Podolsky, *PR* 37, 780 (1931).

Applying the previous analysis to this historic debate, calculating the estimate errors taking into account the correlations between $\hat{x}(0)$, $\hat{x}(t)$ and $\hat{p}(0)$, $\hat{p}(t)$ gives:

$$\Delta[\hat{x}(t) - \hat{x}(0)]^2 \rangle = \langle \Delta \hat{p}^2(0) \rangle\, t^2/m^2$$

$$\langle \Delta[\hat{x}(t) - \hat{p}(0)\, t/m\ ]^2 \rangle = \langle \Delta \hat{x}^2(0) \rangle$$

→ The determination of both position and momentum of a single quantum system at any given time is always limited by the uncertainty principle, even when this time belongs to the past.

→ To this end, quantum mechanics is complete.

Monitoring the Slowly-Varying Quadrature Amplitudes of a Driven Harmonic Oscillator

$$\hat{a}_1(t) - \hat{a}_1(0) \;=\; \delta_1(t) \;=\; \int_0^t dt' \sin(\omega t') f(t')$$

$$\hat{a}_2(t) - \hat{a}_2(0) \;=\; \delta_2(t) \;=\; \int_0^t dt' \cos(\omega t') f(t')$$

Force detection is limited only by the uncertainties in the initial and final simultaneous measurements of the two conjugate quadrature amplitudes:

$$\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle \;\geq\; 1/16$$

$$2\hbar\omega \left(\langle \Delta \hat{a}_1^2 \rangle + \langle \Delta \hat{a}_2^2 \rangle \right) \;\geq\; \hbar\omega$$

→ This limit requires an exchange of at least one quantum of energy between the force and the oscillator per sampling time interval.

Monitoring the Number
of Quanta of Energy

$$\hat{n}(t) - \hat{n}(0) = |\delta(t)|^2 + 2[\delta_1(t)\,\hat{a}_1(0) + \delta_2(t)\,\hat{a}_2(0)]$$

Hollenhorst, *PRD* <u>19</u>, 1669 (1979);
Braginsky, Vorontsov & Thorne, *Science* <u>209</u>, 547 (1980);
Caves, Thorne, Drever, Sandberg & Zimmermann, *RMP* <u>52</u>, 341 (1980):

When the oscillator is initially in a number eigenstate $|k\rangle$, the sensitivity of force detection increases with k.

However, regardless of the initial oscillator state, the uncertainty in the number change and the average number change satisfy

$$\langle \Delta[\hat{n}(t) - \hat{n}(0)]^2 \rangle \geq \langle \hat{n}(t) - \hat{n}(0) \rangle = |\delta(t)|^2$$

where the minimum uncertainty is achieved when the oscillator is initially in the vacuum state $|0\rangle$.

→ This limit requires an exchange of at least one quantum of energy between the force and the oscillator per sampling time interval.

The minimum uncertainty in the number change is due to arbitrary phase between the force and the oscillator, which is also at the root of the quantum Zeno effect of a single oscillator.

Conclusions

There is a fundamental quantum limit to external force detection with a single harmonic oscillator.

This limit is equivalent to the impossibility of determining the quantum state of a single system, and to the quantum Zeno effect of a single system.

This limit requires an exchange of at least one quantum of energy between the external force and the harmonic oscillator per sampling time interval (and vanishes for the free mass).

Force detection beyond this limit is impossible, no matter what quantum state the oscillator is prepared in, what observables of the oscillator are being monitored or what measurement schemes are being employed.
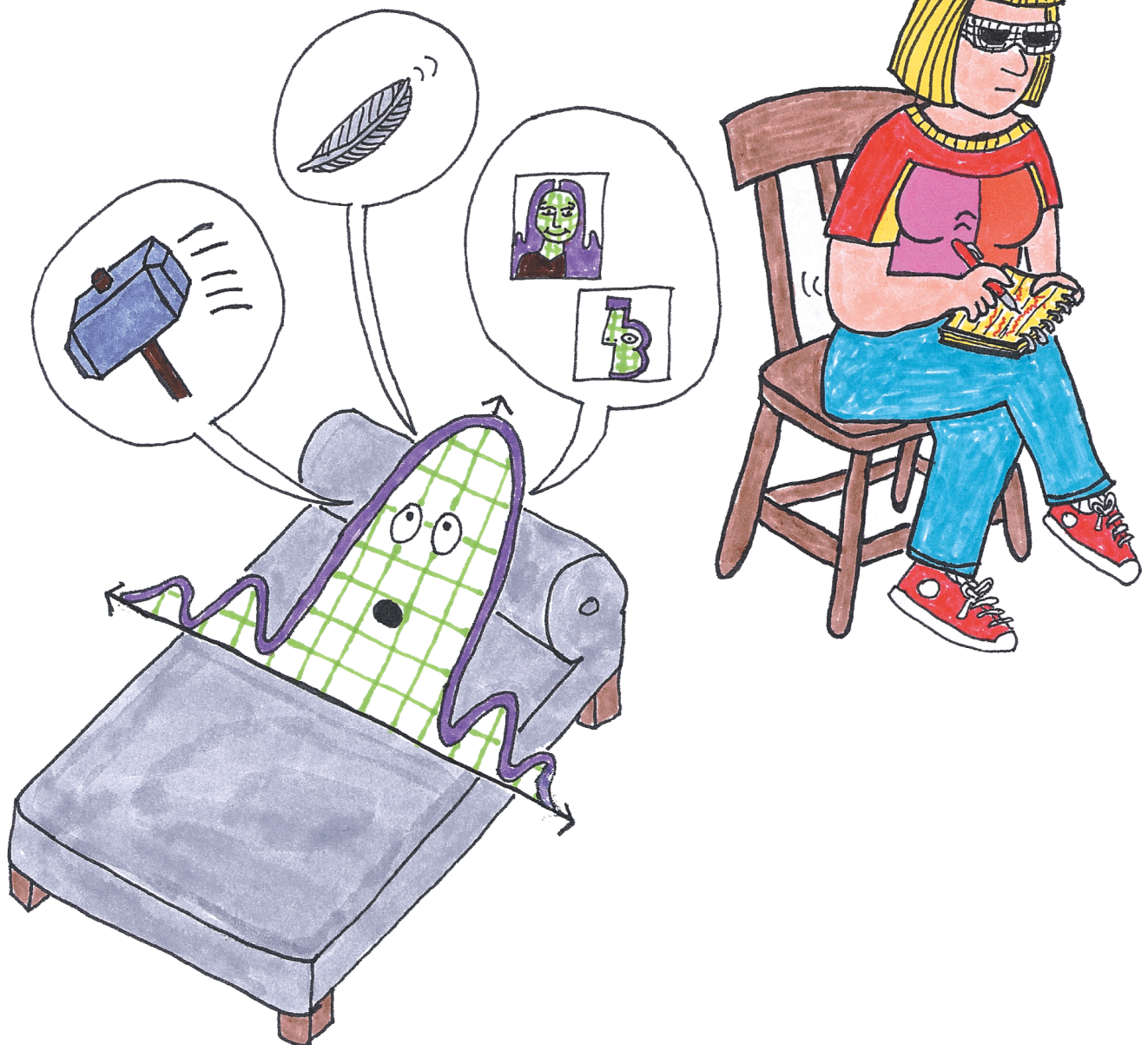
This limit can be achieved via simultaneous monitoring of the time evolution of the two slowly-varying quadrature amplitudes, where the quantum uncertainties associated with the initial oscillator state do not limit the detection of both magnitude and phase.

Determination of the magnitude at this limit can be achieved via monitoring the number of quanta of energy of the oscillator, using either QND or destructive number measurements, where the oscillator state is reset to the vacuum state after each measurement.

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Thank you!